Deriving an Expression for $P(X(t) = x)$ Under the Pareto/NBD Model

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1 Introduction

The Pareto/NBD (Schmittlein et al. 1987, hereafter SMC) is a model for customer-base analysis in a noncontractual setting. One result presented in SMC is an expression for $P(X(t) = x)$, where the random variable $X(t)$ denotes the number of transactions observed in the time interval $(0, t]$. This note derives an alternative expression for this quantity, one that is simpler to evaluate.

In Section 2 we review the assumptions underlying the Pareto/NBD model. In Section 3, we derive an expression for $P(X(t) = x)$ conditional on the unobserved latent characteristics λ and μ. We remove this conditioning in Section 4. For the sake of completeness, SMC's derivation is replicated in the Appendix.

2 Model Assumptions

The Pareto/NBD model is based on the following assumptions:

i. Customers go through two stages in their "lifetime" with a specific firm: they are "alive" for some period of time, then become permanently inactive (i.e., "die").

ii. While alive, the number of transactions made by a customer follows a Poisson process with transaction rate λ. Denoting the number of transactions in the time interval $(0, t]$ by the random variable $X(t)$, it follows that

$$P(X(t) = x \mid \lambda, \text{alive at } t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \dots .$$

iii. A customer's unobserved lifetime of length ω (after which he is viewed as being dead) is exponentially distributed with dropout rate μ:

$$f(\omega \mid \mu) = \mu e^{-\mu\omega} .$$

iv. Heterogeneity in transaction rates across customers follows a gamma distribution with shape parameter r and scale parameter α:

$$g(\lambda \mid r, \alpha) = \frac{\alpha^r \lambda^{r-1} e^{-\lambda\alpha}}{\Gamma(r)} . \tag{1}$$

v. Heterogeneity in dropout rates across customers follows a gamma distribution with shape parameter s and scale parameter β.

$$g(\mu \mid s, \beta) = \frac{\beta^s \mu^{s-1} e^{-\mu\beta}}{\Gamma(s)}. \tag{2}$$

vi. The transaction rate λ and the dropout rate μ vary independently across customers.

3 $P(X(t) = x)$ Conditional on λ and μ

Suppose we know an individual's latent characteristics λ and μ. Assuming the customer was alive at time 0, there are two ways x purchases could have occurred in the interval $(0, t]$:

i. The individual remained alive through the whole interval; this occurs with probability $e^{-\mu t}$. The probability of the individual making x purchases, given he was alive during the whole interval, is $(\lambda t)^x e^{-\lambda t}/x!$. Therefore, the probability of remaining alive through the interval $(0, t]$ and making x purchases is

$$\frac{(\lambda t)^x e^{-(\lambda+\mu)t}}{x!}.$$

ii. The individual died at some point ω ($< t$) and made x purchases in the interval $(0, \omega]$. The probability of this occurring is

$$\begin{aligned}\int_0^t \frac{(\lambda\omega)^x e^{-\lambda\omega}}{x!} \mu e^{-\mu\omega}\, d\omega &= \lambda^x \mu \int_0^t \frac{\omega^x e^{-(\lambda+\mu)\omega}}{x!}\, d\omega \\ &= \frac{\lambda^x \mu}{(\lambda+\mu)^{x+1}} \int_0^t \frac{(\lambda+\mu)^{x+1} \omega^x e^{-(\lambda+\mu)\omega}}{x!}\, d\omega\end{aligned}$$

which, noting that the integrand is an Erlang-$(x+1)$ pdf,

$$= \frac{\lambda^x \mu}{(\lambda+\mu)^{x+1}} \left[1 - e^{-(\lambda+\mu)t} \sum_{i=0}^{x} \frac{\big[(\lambda+\mu)t\big]^i}{i!}\right].$$

Combining these two scenarios gives us the following expression for the probability of observing x purchases in the interval $(0, t]$, conditional on λ and μ:

$$P(X(t) = x \mid \lambda, \mu) = \frac{(\lambda t)^x e^{-(\lambda+\mu)t}}{x!} + \frac{\lambda^x \mu}{(\lambda+\mu)^{x+1}} \left[1 - e^{-(\lambda+\mu)t} \sum_{i=0}^{x} \frac{\big[(\lambda+\mu)t\big]^i}{i!}\right]. \tag{3}$$

4 Removing the Conditioning on λ and μ

In reality, we never know an individual's latent characteristics; we therefore remove the conditioning on λ and μ by taking the expectation of (3) over the distributions of Λ and M:

$$P(X(t) = x \mid r, \alpha, s, \beta) = \int_0^\infty \int_0^\infty P(X(t) = x \mid \lambda, \mu) g(\lambda \mid r, \alpha) g(\mu \mid s, \beta)\, d\lambda\, d\mu. \tag{4}$$

Substituting (1)–(3) in (4) give us

$$P(X(t) = x \mid r, \alpha, s, \beta) = \mathsf{A}_1 + \mathsf{A}_2 - \sum_{i=0}^{x} \frac{t^i}{i!} \mathsf{A}_{3i} \tag{5}$$

where

$$\mathsf{A}_1 = \int_0^\infty \int_0^\infty \frac{(\lambda t)^x e^{-(\lambda+\mu)t}}{x!} g(\lambda \,|\, r, \alpha) g(\mu \,|\, s, \beta)\, d\lambda\, d\mu \tag{6}$$

$$\mathsf{A}_2 = \int_0^\infty \int_0^\infty \frac{\lambda^x \mu}{(\lambda+\mu)^{x+1}} g(\lambda \,|\, r, \alpha) g(\mu \,|\, s, \beta)\, d\lambda\, d\mu \tag{7}$$

$$\mathsf{A}_{3i} = \int_0^\infty \int_0^\infty \frac{\lambda^x \mu e^{-(\lambda+\mu)t}}{(\lambda+\mu)^{x-i+1}} g(\lambda \,|\, r, \alpha) g(\mu \,|\, s, \beta)\, d\lambda\, d\mu \tag{8}$$

i. Solving (6) is trivial:

$$\mathsf{A}_1 = \frac{\Gamma(r+x)}{\Gamma(r)x!} \Big(\frac{\alpha}{\alpha+t}\Big)^r \Big(\frac{t}{\alpha+t}\Big)^x \Big(\frac{\beta}{\beta+t}\Big)^s \tag{9}$$

ii. To solve (7), consider the transformation $Y = M/(\Lambda + M)$ and $Z = \Lambda + M$. Using the transformation technique (Casella and Berger 2002, Section 4.3, pp. 156–162; Mood et al. 1974, Section 6.2, p. 204ff), it follows that the joint distribution of Y and Z is

$$g(y, z \,|\, \alpha, \beta, r, s) = \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} y^{s-1} (1-y)^{r-1} z^{r+s-1} e^{-z(\alpha-(\alpha-\beta)y)} \,. \tag{10}$$

Noting that the inverse of this transformation is $\lambda = (1-y)z$ and $\mu = yz$, it follows that

$$\begin{aligned}
\mathsf{A}_2 &= \int_0^1 \int_0^\infty y(1-y)^x g(y, z \,|\, \alpha, \beta, r, s)\, dz\, dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 \int_0^\infty y^s (1-y)^{r+x-1} z^{r+s-1} e^{-z(\alpha-(\alpha-\beta)y)}\, dz\, dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 y^s (1-y)^{r+x-1} \left\{ \int_0^\infty z^{r+s-1} e^{-z(\alpha-(\alpha-\beta)y)}\, dz \right\} dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \Gamma(r+s) \int_0^1 y^s (1-y)^{r+x-1} (\alpha - (\alpha-\beta)y)^{-(r+s)}\, dy \\
&= \frac{\alpha^r \beta^s}{B(r,s)} \frac{1}{\alpha^{r+s}} \int_0^1 y^s (1-y)^{r+x-1} \big[1 - \big(\tfrac{\alpha-\beta}{\alpha}\big)y\big]^{-(r+s)}\, dy
\end{aligned}$$

which, recalling Euler's integral for the Gaussian hypergeometric function,[1]

$$= \frac{\alpha^r \beta^s}{\alpha^{r+s}} \frac{B(r+x, s+1)}{B(r,s)} {}_2F_1\big(r+s, s+1; r+s+x+1; \tfrac{\alpha-\beta}{\alpha}\big) \,. \tag{11}$$

Looking closely at (11), we see that the argument of the Gaussian hypergeometric function, $\frac{\alpha-\beta}{\alpha}$, is guaranteed to be bounded between 0 and 1 when $\alpha \ge \beta$ (since $\beta > 0$), thus ensuring convergence of the series representation of the function. However, when $\alpha < \beta$ we can be faced with the situation where $\frac{\alpha-\beta}{\alpha} < -1$, in which case the series is divergent.

Applying the linear transformation (Abramowitz and Stegun 1972, equation 15.3.4)

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\big(a, c-b; c; \tfrac{z}{z-1}\big) \,, \tag{12}$$

gives us

$$\mathsf{A}_2 = \frac{\alpha^r \beta^s}{\beta^{r+s}} \frac{B(r+x, s+1)}{B(r,s)} {}_2F_1\big(r+s, r+x; r+s+x+1; \tfrac{\beta-\alpha}{\beta}\big) \,. \tag{13}$$

We see that the argument of the above Gaussian hypergeometric function is bounded between 0 and 1 when $\alpha \le \beta$. We therefore present (11) and (13) as solutions to (7): we use (11) when $\alpha \ge \beta$ and (13) when $\alpha \le \beta$.

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[1] ${}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \,, \; c > b \,.$

iii. To solve (8), we also make use of the transformation $Y = M/(\Lambda + M)$ and $Z = \Lambda + M$. Given (10), it follows that

$$\begin{aligned}
\mathsf{A}_{3i} &= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 \int_0^\infty y^s (1-y)^{r+x-1} z^{r+s+i-1} e^{-z(\alpha+t-(\alpha-\beta)y)}\, dz\, dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 y^s (1-y)^{r+x-1} \left\{ \int_0^\infty z^{r+s+i-1} e^{-z(\alpha+t-(\alpha-\beta)y)}\, dz \right\} dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \Gamma(r+s+i) \int_0^1 y^s (1-y)^{r+x-1} (\alpha + t - (\alpha-\beta)y)^{-(r+s+i)}\, dy \\
&= \frac{\alpha^r \beta^s}{B(r,s)} \frac{\Gamma(r+s+i)}{\Gamma(r+s)} \frac{1}{(\alpha+t)^{r+s+i}} \int_0^1 y^s (1-y)^{r+x-1} \big[1 - \big(\tfrac{\alpha-\beta}{\alpha+t}\big) y\big]^{-(r+s+i)}\, dy
\end{aligned}$$

which, recalling Euler's integral for the Gaussian hypergeometric function,

$$\begin{aligned}
&= \frac{\alpha^r \beta^s}{(\alpha+t)^{r+s+i}} \frac{B(r+x, s+1)}{B(r,s)} \frac{\Gamma(r+s+i)}{\Gamma(r+s)} \\
&\qquad \times {}_2F_1\big(r+s+i, s+1; r+s+x+1; \tfrac{\alpha-\beta}{\alpha+t}\big)\,. \qquad (14)
\end{aligned}$$

Noting that the argument of the Gaussian hypergeometric function is only guaranteed to be bounded between 0 and 1 when $\alpha \geq \beta$, we apply the linear transformation (12), which gives us

$$\begin{aligned}
\mathsf{A}_{3i} = &\frac{\alpha^r \beta^s}{(\beta+t)^{r+s+i}} \frac{B(r+x, s+1)}{B(r,s)} \frac{\Gamma(r+s+i)}{\Gamma(r+s)} \\
&\qquad \times {}_2F_1\big(r+s+i, r+x; r+s+x+1; \tfrac{\beta-\alpha}{\beta+t}\big)\,. \qquad (15)
\end{aligned}$$

The argument of the above Gaussian hypergeometric function is bounded between 0 and 1 when $\alpha \leq \beta$. We therefore present (14) and (15) as solutions to (8), using (14) when $\alpha \geq \beta$ and (15) when $\alpha \leq \beta$.

Substituting (9), (11), (13), (14), and (15) in (5) yields the following expression for the distribution of the number of transactions in the interval $(0, t]$ for a randomly-chosen individual under the Pareto/NBD model:

$$\begin{aligned}
P(X(t) = x \,|\, r, \alpha, s, \beta) = &\frac{\Gamma(r+x)}{\Gamma(r)x!} \Big(\frac{\alpha}{\alpha+t}\Big)^r \Big(\frac{t}{\alpha+t}\Big)^x \Big(\frac{\beta}{\beta+t}\Big)^s \\
&+ \alpha^r \beta^s \frac{B(r+x, s+1)}{B(r,s)} \Big\{ \mathsf{B}_1 - \sum_{i=0}^{x} \frac{\Gamma(r+s+i)}{\Gamma(r+s)} \frac{t^i}{i!} \mathsf{B}_{2i} \Big\} \qquad (16)
\end{aligned}$$

where

$$\mathsf{B}_1 = \begin{cases} \dfrac{{}_2F_1\big(r+s, s+1; r+s+x+1; \frac{\alpha-\beta}{\alpha}\big)}{\alpha^{r+s}} & \text{if } \alpha \geq \beta \\[2ex] \dfrac{{}_2F_1\big(r+s, r+x; r+s+x+1; \frac{\beta-\alpha}{\beta}\big)}{\beta^{r+s}} & \text{if } \alpha \leq \beta \end{cases}$$

and

$$\mathsf{B}_{2i} = \begin{cases} \dfrac{{}_2F_1\big(r+s+i, s+1; r+s+x+1; \frac{\alpha-\beta}{\alpha+t}\big)}{(\alpha+t)^{r+s+i}} & \text{if } \alpha \geq \beta \\[2ex] \dfrac{{}_2F_1\big(r+s+i, r+x; r+s+x+1; \frac{\beta-\alpha}{\beta+t}\big)}{(\beta+t)^{r+s+i}} & \text{if } \alpha \leq \beta\,. \end{cases}$$

We note that this expression requires $x+2$ evaluations of the Gaussian hypergeometric function. In contrast, SMC's expression (see the attached appendix) requires $2(x+1)$ evaluations of the Gaussian hypergeometric function.

The equivalence of (16) and (A1), (A3), (A4) is not immediately obvious. Purely from a logical perspective, they must be equivalent. Furthermore, equivalence is observed in numerical investigations. However, we have yet to demonstrate direct equivalence of these two expressions for $P(X(t) = x \,|\, r, \alpha, s, \beta)$. Stay tuned.

References

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Appendix: SMC's Derivation of $P(X(t) = x \mid r, \alpha, s, \beta)$

SMC derive their expression for $P(X(t) = x)$ by first integrating over λ and μ and then removing the conditioning on ω, which is the reverse of the approach used in Sections 3 and 4 above. This gives us

$$\begin{aligned} P(X(t) = x \mid r, \alpha, s, \beta) &= \underbrace{\frac{\Gamma(r+x)}{\Gamma(r)x!}\Big(\frac{\alpha}{\alpha+t}\Big)^r\Big(\frac{t}{\alpha+t}\Big)^x}_{\text{NBD } P(X(t)=x)}\underbrace{\Big(\frac{\beta}{\beta+t}\Big)^s}_{P(\Omega>t)} \\ &\quad + \int_0^t \underbrace{\frac{\Gamma(r+x)}{\Gamma(r)x!}\Big(\frac{\alpha}{\alpha+\omega}\Big)^r\Big(\frac{\omega}{\alpha+\omega}\Big)^x}_{\text{NBD } P(X(\omega)=x)}\underbrace{\frac{s}{\beta}\Big(\frac{\beta}{\beta+\omega}\Big)^{s+1}}_{f(\omega)} d\omega \\ &= \frac{\Gamma(r+x)}{\Gamma(r)x!}\Big(\frac{\alpha}{\alpha+t}\Big)^r\Big(\frac{t}{\alpha+t}\Big)^x\Big(\frac{\beta}{\beta+t}\Big)^s + \frac{\Gamma(r+x)}{\Gamma(r)x!} s\alpha^r\beta^s \, \mathsf{C} \end{aligned} \tag{A1}$$

where

$$\mathsf{C} = \int_0^t \omega^x(\alpha+\omega)^{-(r+x)}(\beta+\omega)^{-(s+1)} d\omega\,. \tag{A2}$$

Making the change of variable $y = \alpha + \omega$,

$$\mathsf{C} = \int_\alpha^{\alpha+t} (y-\alpha)^x y^{-(r+x)}(y-\alpha+\beta)^{-(s+1)} dy$$

which, recalling the binomial theorem,[2]

$$\begin{aligned} &= \int_\alpha^{\alpha+t} \Big\{\sum_{j=0}^x \binom{x}{j} y^{x-j}(-\alpha)^j\Big\} y^{-(r+x)}(y-\alpha+\beta)^{-(s+1)} dy \\ &= \sum_{j=0}^x \binom{x}{j}(-\alpha)^j \int_\alpha^{\alpha+t} y^{-(r+j)}(y-\alpha+\beta)^{-(s+1)} dy \\ &= \sum_{j=0}^x \binom{x}{j}(-\alpha)^j \Big\{\int_\alpha^\infty y^{-(r+j)}(y-\alpha+\beta)^{-(s+1)} dy - \int_{\alpha+t}^\infty y^{-(r+j)}(y-\alpha+\beta)^{-(s+1)} dy\Big\} \end{aligned}$$

letting $z = \alpha/y$ in the first integral (which implies $dy = -dz\alpha z^{-2}$) and $z = (\alpha+t)/y$ in the second integral (which implies $dy = -dz(\alpha+t)z^{-2}$),

$$\begin{aligned} &= \sum_{j=0}^x \binom{x}{j}(-\alpha)^j \Big\{\alpha^{-(r+s+j)} \int_0^1 z^{r+s+j-1}\big(1 - \tfrac{\alpha-\beta}{\alpha} z\big)^{-(s+1)} dz \\ &\qquad - (\alpha+t)^{-(r+s+j)} \int_0^1 z^{r+s+j-1}\big(1 - \tfrac{\alpha-\beta}{\alpha+t} z\big)^{-(s+1)} dz\Big\} \end{aligned}$$

which, recalling Euler's integral for the Gaussian hypergeometric function,

$$\begin{aligned} &= \sum_{j=0}^x \binom{x}{j}\frac{(-\alpha)^j}{r+s+j}\Big\{\frac{{}_2F_1\big(s+1, r+s+j; r+s+j+1; \frac{\alpha-\beta}{\alpha}\big)}{\alpha^{r+s+j}} \\ &\qquad - \frac{{}_2F_1\big(s+1, r+s+j; r+s+j+1; \frac{\alpha-\beta}{\alpha+t}\big)}{(\alpha+t)^{r+s+j}}\Big\}\,. \end{aligned} \tag{A3}$$

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[2] $(x+y)^r = \sum_{j=0}^x \binom{r}{k} x^k y^{r-k}$ for integer $r \geq 0$.

We note that the arguments of the above Gaussian hypergeometric functions are only guaranteed to be bounded between 0 and 1 when $\alpha \geq \beta$. We therefore revisit (A2), applying the change of variable $y = \beta + \omega$:

$$\mathsf{C} = \int_{\beta}^{\beta+t} (y-\beta)^x y^{-(s+1)} (y-\beta+\alpha)^{-(r+x)} dy$$

which, recalling the binomial theorem,

$$\begin{aligned}
&= \int_{\beta}^{\beta+t} \left\{ \sum_{j=0}^{x} \binom{x}{j} y^{x-j} (-\beta)^j \right\} y^{-(s+1)} (y-\beta+\alpha)^{-(r+x)} dy \\
&= \sum_{j=0}^{x} \binom{x}{j} (-\beta)^j \int_{\beta}^{\beta+t} y^{-(s+j+1-x)} (y-\beta+\alpha)^{-(r+x)} dy \\
&= \sum_{j=0}^{x} \binom{x}{j} (-\beta)^j \left\{ \int_{\beta}^{\infty} y^{-(s+j+1-x)} (y-\beta+\alpha)^{-(r+x)} dy \right. \\
&\qquad\qquad \left. - \int_{\beta+t}^{\infty} y^{-(s+j+1-x)} (y-\beta+\alpha)^{-(r+x)} dy \right\}
\end{aligned}$$

letting $z = \beta/y$ in the first integral (which implies $dy = -dz\beta z^{-2}$) and $z = (\beta+t)/y$ in the second integral (which implies $dy = -dz(\beta+t)z^{-2}$),

$$\begin{aligned}
&= \sum_{j=0}^{x} \binom{x}{j} (-\beta)^j \left\{ \beta^{-(r+s+j)} \int_0^1 z^{r+s+j-1} \left(1 - \tfrac{\beta-\alpha}{\beta} z\right)^{-(r+x)} dz \right. \\
&\qquad\qquad \left. - (\beta+t)^{-(r+s+j)} \int_0^1 z^{r+s+j-1} \left(1 - \tfrac{\beta-\alpha}{\beta+t} z\right)^{-(r+x)} dz \right\}
\end{aligned}$$

which, recalling Euler's integral for the Gaussian hypergeometric function,

$$\begin{aligned}
&= \sum_{j=0}^{x} \binom{x}{j} \frac{(-\beta)^j}{r+s+j} \left\{ \frac{{}_2F_1\left(r+x, r+s+j; r+s+j+1; \frac{\beta-\alpha}{\beta}\right)}{\beta^{r+s+j}} \right. \\
&\qquad\qquad \left. - \frac{{}_2F_1\left(r+x, r+s+j; r+s+j+1; \frac{\beta-\alpha}{\beta+t}\right)}{(\beta+t)^{r+s+j}} \right\}. \qquad \text{(A4)}
\end{aligned}$$

We note that (A3) and (A4) each require $2(x+1)$ evaluations of the Gaussian hypergeometric function.