An Introduction to Probability Models for Marketing Research

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Problem 1:
Projecting Customer Retention Rates

(Modelling Discrete-Time Duration Data)

Background

One of the most important problems facing marketing managers today is the issue of *customer retention*. It is vitally important for firms to be able to anticipate the number of customers who will remain active for $1, 2, \ldots, T$ periods (e.g., years or months) after they are first acquired by the firm.

The following dataset is taken from a popular book on data mining (Berry and Linoff, *Data Mining Techniques*, Wiley 2004). It documents the “survival” pattern over a seven-year period for a sample of customers who were all “acquired” in the same period.

Customers Surviving At Least 0–7 Years

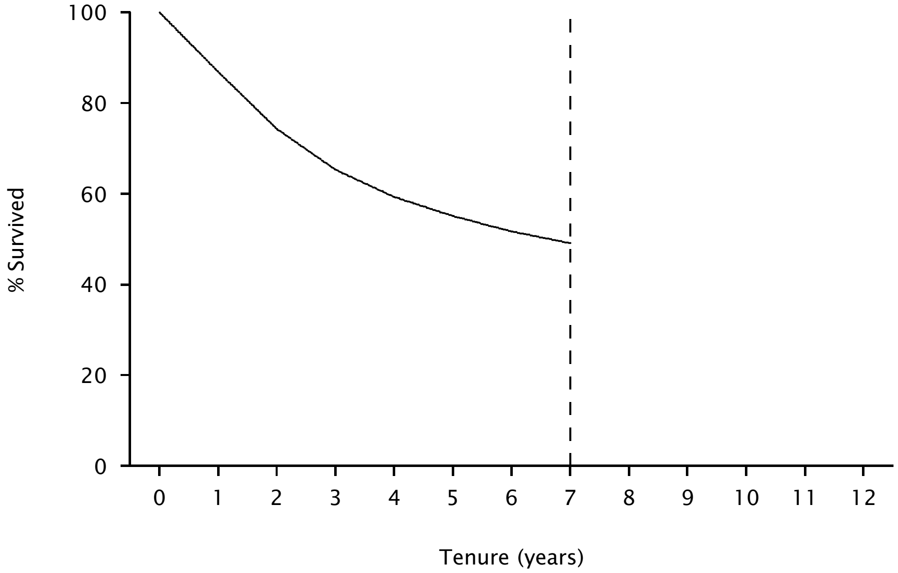
Year	# Customers	% Alive
0	1000	100.0%
1	869	86.9%
2	743	74.3%
3	653	65.3%
4	593	59.3%
5	551	55.1%
6	517	51.7%
7	491	49.1%

Of the 1000 initial customers, 869 renew their contracts at the end of the first year. At the end of the second year, 743 of these 869 customers renew their contracts.

Modelling Objective

Develop a model that enables us to project the survival curve (and therefore retention rates) over the next five years (i.e., out to $T = 12$).

Modeling Objective



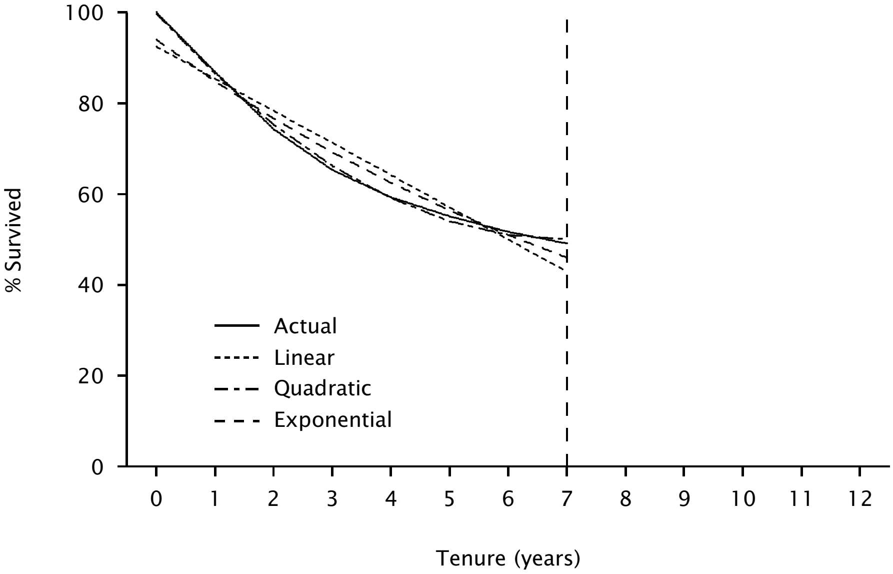
Natural Starting Point

Project survival using simple functions of time:

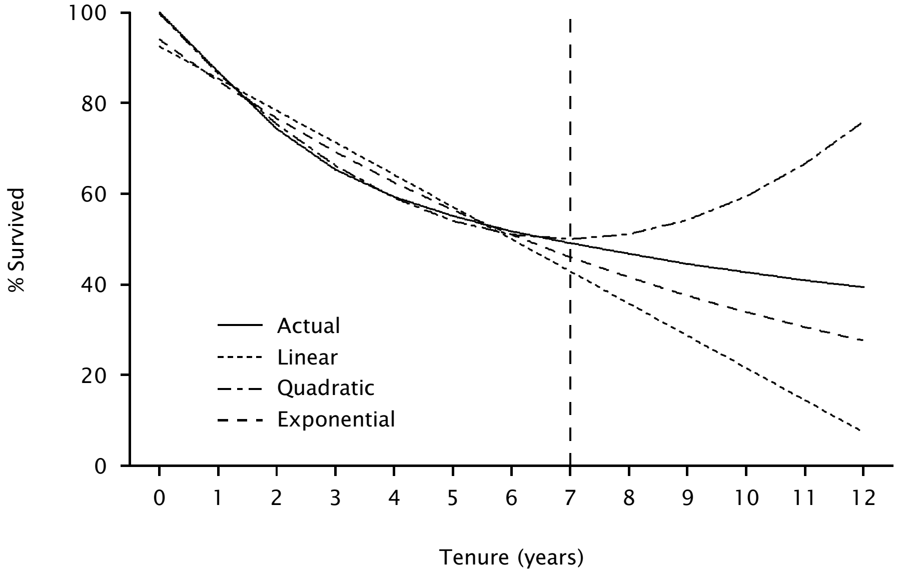
- Consider linear, quadratic, and exponential functions
- Let y = the proportion of customers surviving at least t years

$$\begin{aligned} y &= 0.925 - 0.071t & R^2 &= 0.922 \\ y &= 0.997 - 0.142t + 0.010t^2 & R^2 &= 0.998 \\ \ln(y) &= -0.062 - 0.102t & R^2 &= 0.964 \end{aligned}$$

Model Fit



Survival Curve Projections



Developing a Better Model (I)

Consider the following story of customer behavior:

i. At the end of each period, an individual renews his contract with (constant and unobserved) probability $1 - \theta$.

ii. All customers have the same “churn probability” θ.

Developing a Better Model (I)

More formally:

- Let the random variable T denote the duration of the customer's relationship with the firm.
- We assume that the random variable T has a (shifted) geometric distribution with parameter θ:

$$\begin{aligned}
P(T = t \mid \theta) &= \theta(1-\theta)^{t-1}, \quad t = 1, 2, 3, \ldots \\
P(T > t \mid \theta) &= (1-\theta)^{t}, \quad t = 1, 2, 3, \ldots
\end{aligned}$$

Developing a Better Model (I)

The probability of the observed pattern of contract renewals is:

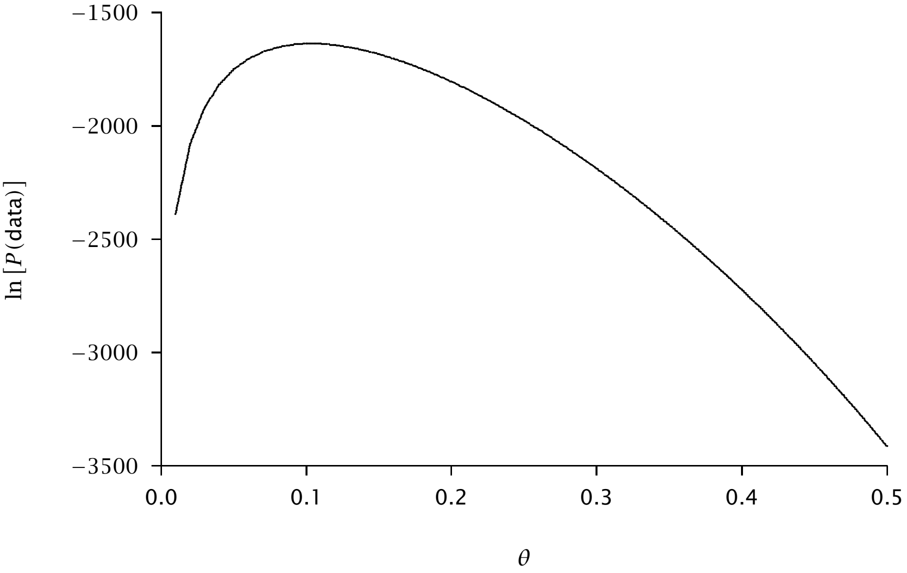
$$\begin{aligned}
&[\theta]^{131}[\theta(1-\theta)^1]^{126}[\theta(1-\theta)^2]^{90} \\
&\quad \times [\theta(1-\theta)^3]^{60}[\theta(1-\theta)^4]^{42}[\theta(1-\theta)^5]^{34} \\
&\qquad \times [\theta(1-\theta)^6]^{26}[(1-\theta)^7]^{491}
\end{aligned}$$

Estimating Model Parameters

- Let us assume that the observed data are the outcome of a process characterized the “coin-flipping” model of contract renewal.
- Which value of θ is more likely to have “generated” the data?

θ	$P(\text{data})$	$\ln[P(\text{data})]$
0.2	1.49×10^{-784}	-1804.8
0.5	1.34×10^{-1483}	-3414.4

Estimating Model Parameters



Estimating Model Parameters

We estimate the model parameters using the method of *maximum likelihood*:

- The likelihood function is defined as the probability of observing the sample data for a given set of the (unknown) model parameters
- This probability is computed using the model and is viewed as a function of the model parameters:

$$L(\text{parameters}|\text{data}) = p(\text{data}|\text{parameters})$$

- For a given dataset, the maximum likelihood estimates of the model parameters are those values that maximize $L(\cdot)$

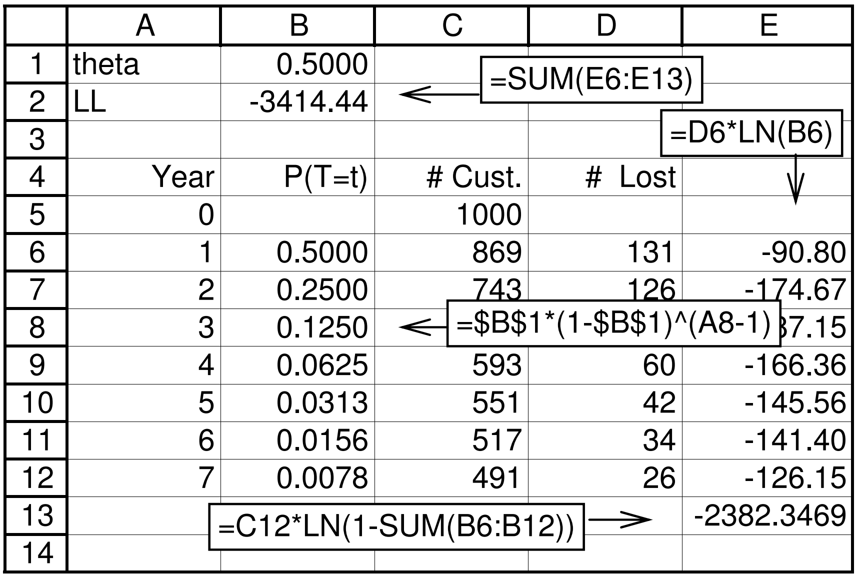
Estimating Model Parameters

The log-likelihood function is defined as:

$$\begin{aligned} LL(\theta|\text{data}) = {} & 131 \times \ln[P(T = 1)] + \\ & 126 \times \ln[P(T = 2)] + \\ & \dots \qquad\qquad + \\ & 26 \times \ln[P(T = 7)] + \\ & 491 \times \ln[P(T > 7)] \end{aligned}$$

The maximum value of the log-likelihood function is $LL = -1637.09$, which occurs at $\hat{\theta} = 0.103$.

Estimating Model Parameters



	A	B	C	D	E
1	theta	0.5000			
2	LL	-3414.44			
3					
4	Year	P(T=t)	# Cust.	# Lost	
5	0		1000		
6	1	0.5000	869	131	-90.80
7	2	0.2500	743	126	-174.67
8	3	0.1250			37.15
9	4	0.0625	593	60	-166.36
10	5	0.0313	551	42	-145.56
11	6	0.0156	517	34	-141.40
12	7	0.0078	491	26	-126.15
13					-2382.3469
14					

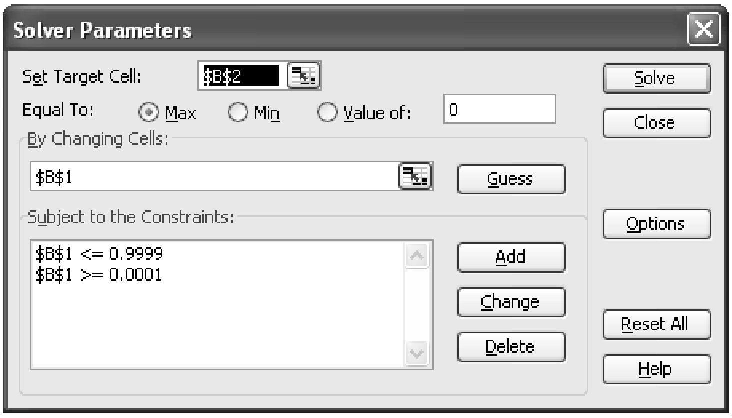
=SUM(E6:E13)

=D6*LN(B6)

=$B$1*(1-$B$1)^(A8-1)

=C12*LN(1-SUM(B6:B12))

Estimating Model Parameters



Set Target Cell: $B$2

Equal To: ● Max ○ Min ○ Value of: 0

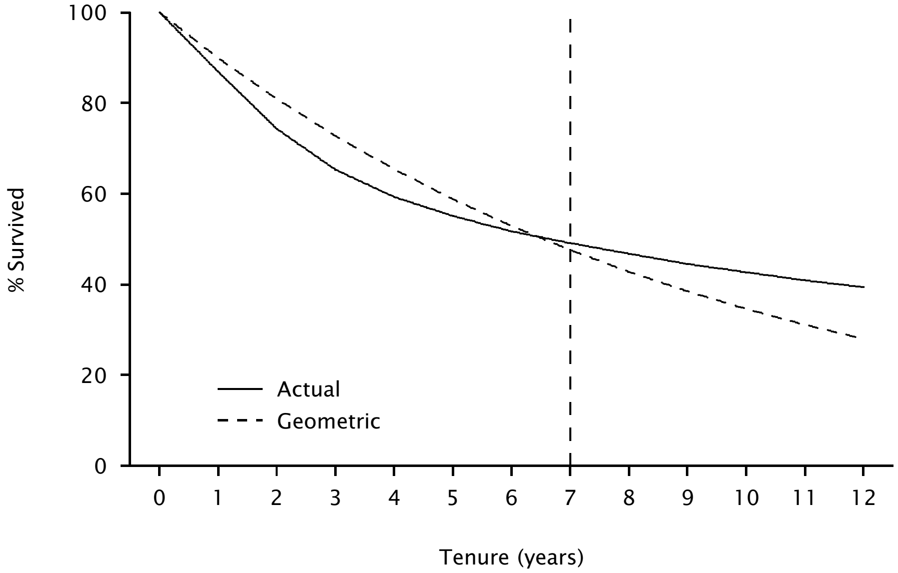
By Changing Cells:

$B$1

Subject to the Constraints:

$B$1 <= 0.9999
$B$1 >= 0.0001

Survival Curve Projection



What’s wrong with this story of customer contract-renewal behavior?

Developing a Better Model (II)

Consider the following story of customer behavior:

i. At the end of each period, an individual renews his contract with (constant and unobserved) probability $1 - \theta$.

ii. "Churn probabilities" vary across customers.

Accounting for Heterogeneity (I)

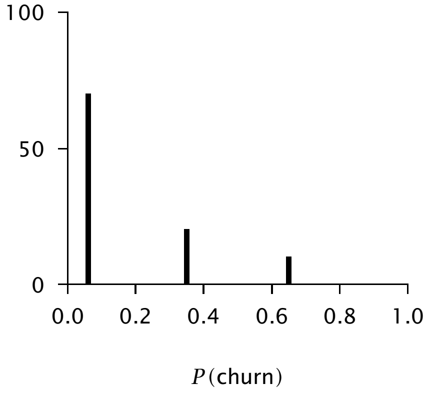
- We don't know each customer's true value of θ.
  $\longrightarrow$ we need to take a weighted average over all possible values that θ can take on.
- If there were only two segments of customers,

$$\begin{aligned} P(T = t) &= P(T = t \mid \text{segment 1})P(\text{segment 1}) \\ &\quad + P(T = t \mid \text{segment 2})P(\text{segment 2}) \\ &= \theta_1(1 - \theta_1)^{t-1}\pi + \theta_2(1 - \theta_2)^{t-1}(1 - \pi) \end{aligned}$$

- Likewise for three or four segments . . .

Vodafone Italia
Churn Clusters

Cluster	$P(\text{churn})$	%CB
Low risk	0.06	70
Medium risk	0.35	20
High risk	0.65	10



Source: “Vodafone Achievement and Challenges in Italy” presentation (2003-09-12)

Accounting for Heterogeneity (II)

- We move from a finite number of segments to an infinite number of segments.
- Assume heterogeneity in θ is captured by a beta distribution with pdf

$$g(\theta \mid \alpha, \beta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}\,.$$

The Beta Function

- The beta function $B(a, b)$ is defined by the integral

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,\ a > 0, b > 0\,,$$

  and can be expressed in terms of gamma functions:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,.$$

- The gamma function $\Gamma(a)$ is defined by the integral

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt,\ a > 0\,,$$

  and has the recursive property $\Gamma(a+1) = a\Gamma(a)$.

The Beta Distribution

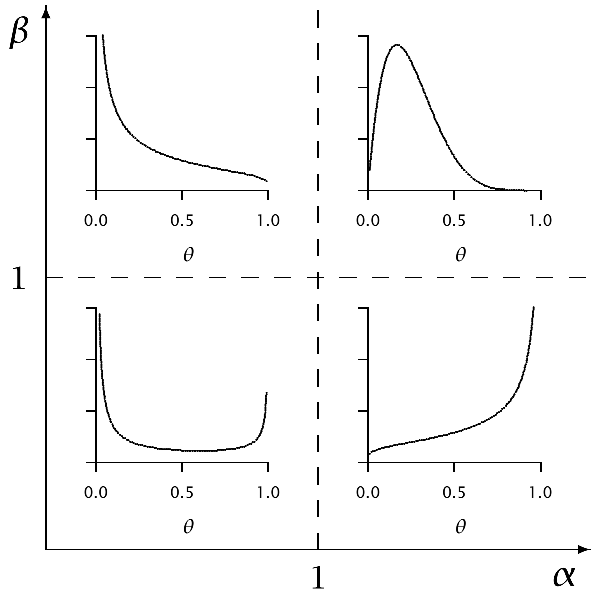
$$g(\theta \mid \alpha, \beta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}\,,\ 0 < \theta < 1\,.$$

- The mean of the beta distribution is

$$E(\Theta) = \frac{\alpha}{\alpha+\beta}$$

- The beta distribution is a flexible distribution … and is mathematically convenient

General Shapes of the Beta Distribution



Developing a Better Model (IIc)

For a randomly chosen individual,

$$\begin{aligned} P(T = t \mid \alpha, \beta) &= \int_0^1 P(T = t \mid \theta) g(\theta \mid \alpha, \beta)\, d\theta \\ &= \frac{B(\alpha + 1, \beta + t - 1)}{B(\alpha, \beta)}\,. \\ P(T > t \mid \alpha, \beta) &= \int_0^1 P(T > t \mid \theta) g(\theta \mid \alpha, \beta)\, d\theta \\ &= \frac{B(\alpha, \beta + t)}{B(\alpha, \beta)}\,. \end{aligned}$$

We call this “continuous mixture” model the shifted-beta-geometric (sBG) distribution

Computing sBG Probabilities

We can compute sBG probabilities by using the following forward-recursion formula from $P(T = 1)$:

$$P(T = t) = \begin{cases} \dfrac{\alpha}{\alpha + \beta} & t = 1 \\ \\ \dfrac{\beta + t - 2}{\alpha + \beta + t - 1} P(T = t - 1) & t = 2, 3, \ldots \end{cases}$$

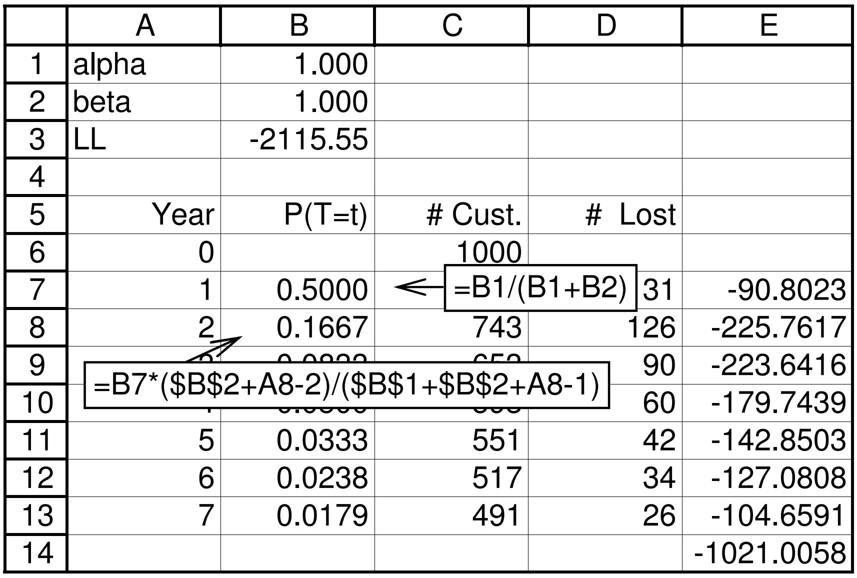
Estimating Model Parameters

The log-likelihood function is defined as:

$$\begin{aligned} LL(\alpha, \beta | \text{data}) = {} & 131 \times \ln[P(T = 1)] + \\ & 126 \times \ln[P(T = 2)] + \\ & \quad \ldots \qquad\qquad + \\ & 26 \times \ln[P(T = 7)] + \\ & 491 \times \ln[P(T > 7)] \end{aligned}$$

The maximum value of the log-likelihood function is $LL = -1611.16$, which occurs at $\hat{\alpha} = 0.668$ and $\hat{\beta} = 3.806$.

Estimating Model Parameters

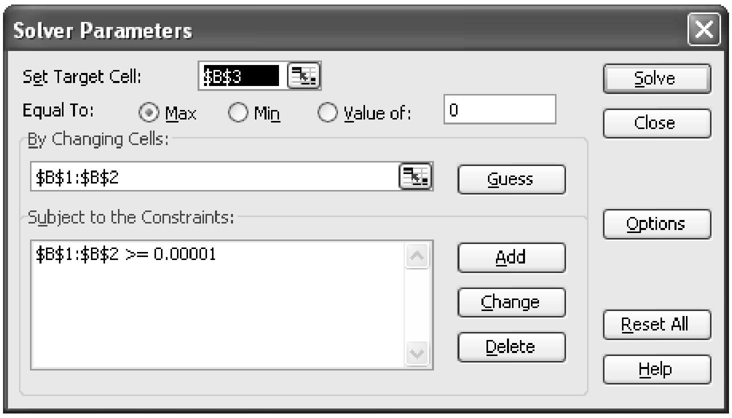


	A	B	C	D	E
1	alpha	1.000			
2	beta	1.000			
3	LL	-2115.55			
4					
5	Year	P(T=t)	# Cust.	# Lost	
6	0		1000		
7	1	0.5000	[illegible]	31	-90.8023
8	2	0.1667	743	126	-225.7617
9	3	0.0833	653	90	-223.6416
10	4	0.0500	593	60	-179.7439
11	5	0.0333	551	42	-142.8503
12	6	0.0238	517	34	-127.0808
13	7	0.0179	491	26	-104.6591
14					-1021.0058

=B1/(B1+B2)

=B7*($B$2+A8-2)/($B$1+$B$2+A8-1)

Estimating Model Parameters



Solver Parameters

Set Target Cell: $B$3

Equal To: ● Max ○ Min ○ Value of: 0

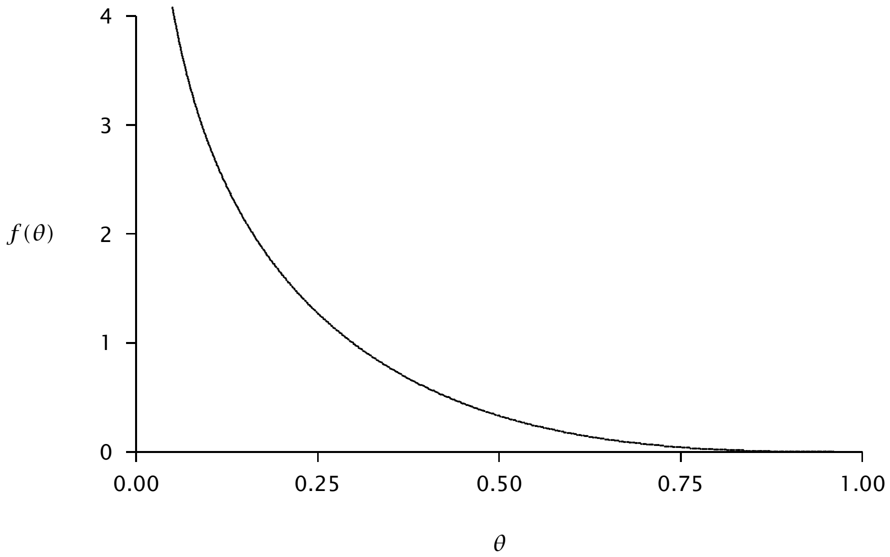
By Changing Cells:

$B$1:$B$2

Subject to the Constraints:

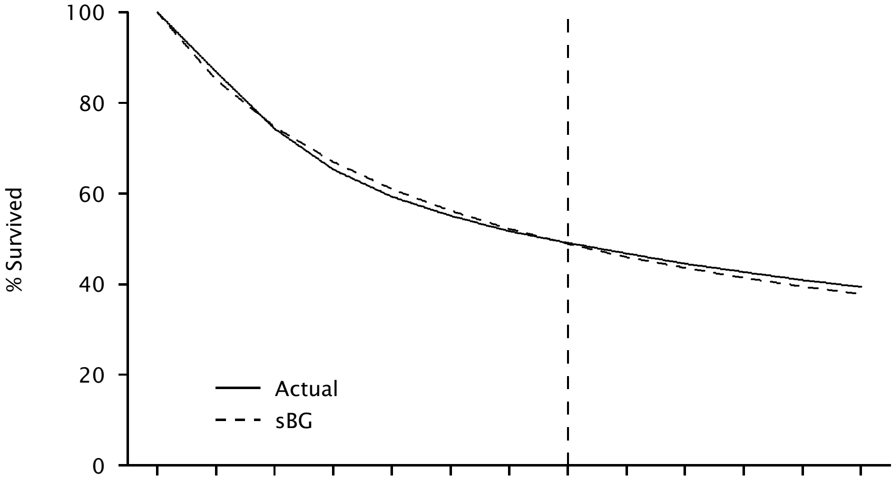
$B$1:$B$2 >= 0.00001

Estimated Distribution of Churn Probabilities



$$\hat{\alpha} = 0.668, \hat{\beta} = 3.806, \widehat{E(\Theta)} = 0.149$$

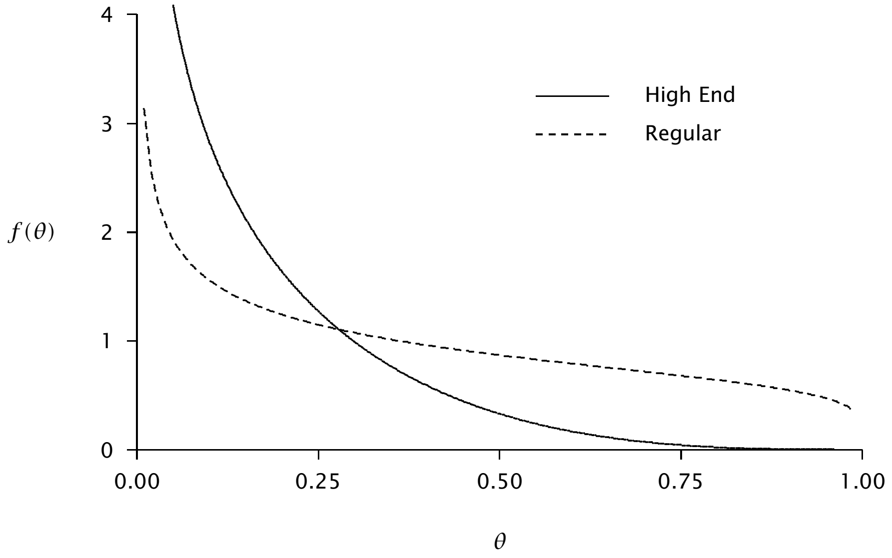
Survival Curve Projection



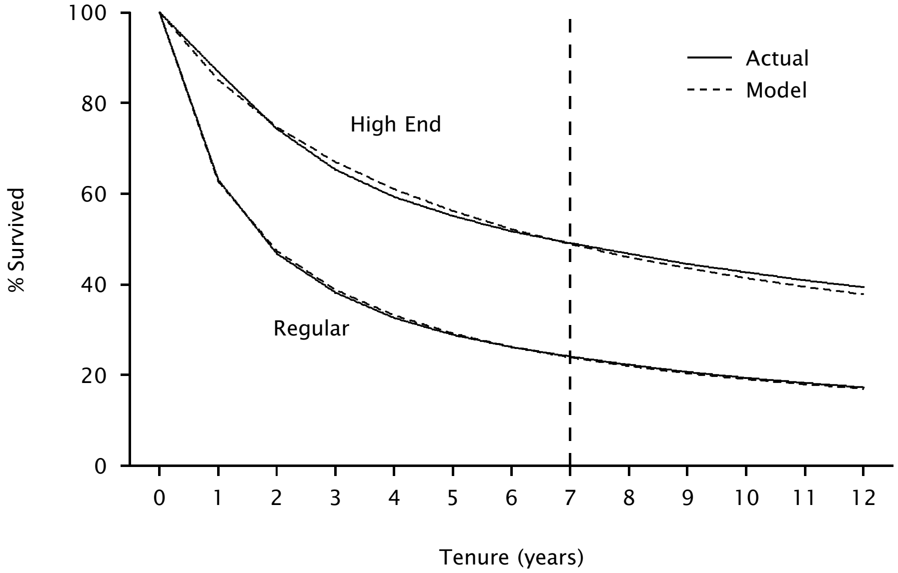
A Further Test of the sBG Model

- The dataset we have been analyzing is for a “high end” segment of customers.
- We also have a dataset for a “regular” customer segment.
- Fitting the sBG model to the data on contract renewals for this segment yields $\hat{\alpha} = 0.704$ and $\hat{\beta} = 1.182$ ($\Longrightarrow \widehat{E(\Theta)} = 0.373$).

Estimated Distributions of Churn Probabilities



Survival Curve Projections



Implied Retention Rates

- The retention rate for period t (r_t) is defined as the proportion of customers who had renewed their contract at the end of period $t - 1$ who then renew their contract at the end of period t.
- For any model of contract duration with survivor function $P(T > t)$,

$$r_t = \frac{P(T > t)}{P(T > t - 1)}$$

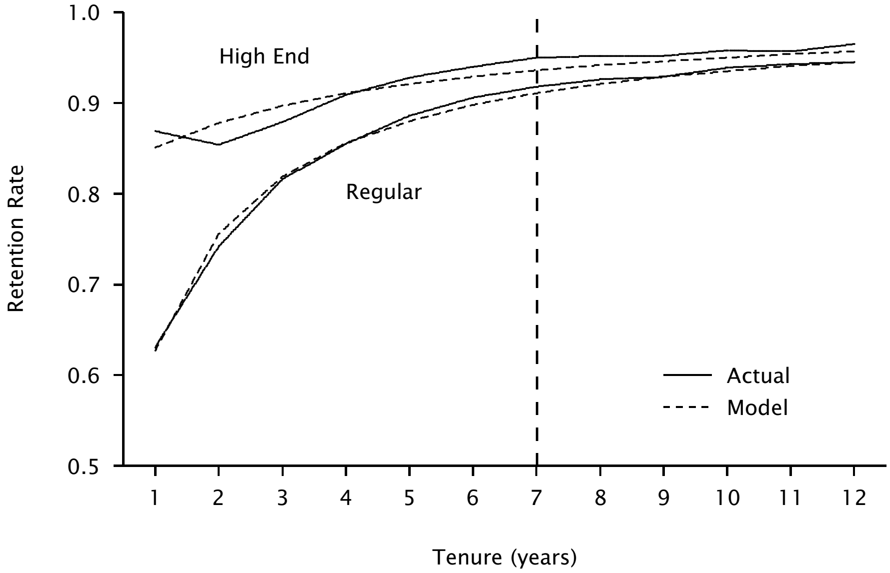
Implied Retention Rates

- For the sBG model,

$$r_t = \frac{\beta + t - 1}{\alpha + \beta + t - 1}$$

- An increasing function of time, even though the individual-level retention probability is constant.
- A sorting effect in a heterogeneous population.

Projecting Retention Rates



Concepts and Tools Introduced

- Probability models
- Maximum-likelihood estimation of model parameters
- Modelling discrete-time (single-event) duration data
- Models of contract renewal behavior

Further Reading

Fader, Peter S. and Bruce G. S. Hardie (2007), “How to Project Customer Retention,” *Journal of Interactive Marketing*, **21** (Winter), 76–90.

Fader, Peter S. and Bruce G. S. Hardie (2007), “How Not to Project Customer Retention.” <`http://brucehardie.com/notes/016/`>

Lee, Ka Lok, Peter S. Fader, and Bruce G.S. Hardie (2007), “How to Project Patient Persistency,” *FORESIGHT*, Issue 8, Fall, 31–35.

Buchanan, Bruce and Donald G. Morrison (1988), “A Stochastic Model of List Falloff with Implications for Repeat Mailings,” *Journal of Direct Marketing*, **2** (Summer), 7–15.

Potter, R. G. and M. P. Parker (1964), “Predicting the Time Required to Conceive,” *Population Studies*, **18** (July), 99–116.

Introduction to Probability Models

The Logic of Probability Models

- Many researchers attempt to describe/predict behavior using observed variables.
- However, they still use random components in recognition that not all factors are included in the model.
- We treat behavior as if it were “random” (probabilistic, stochastic).
- We propose a model of individual-level behavior which is “summed” across individuals (taking individual differences into account) to obtain a model of aggregate behavior.

Uses of Probability Models

- Understanding market-level behavior patterns
- Prediction
  - To settings (e.g., time periods) beyond the observation period
  - Conditional on past behavior
- Profiling behavioral propensities of individuals
- Benchmarks/norms

Building a Probability Model

(i) Determine the marketing decision problem/ information needed.

(ii) Identify the *observable* individual-level behavior of interest.

- We denote this by x.

(iii) Select a probability distribution that characterizes this individual-level behavior.

- This is denoted by $f(x|\theta)$.
- We view the parameters of this distribution as individual-level *latent characteristics*.

Building a Probability Model

(iv) Specify a distribution to characterize the distribution of the latent characteristic variable(s) across the population.

- We denote this by $g(\theta)$.
- This is often called the *mixing distribution*.

(v) Derive the corresponding *aggregate* or *observed* distribution for the behavior of interest:

$$f(x) = \int f(x|\theta)g(\theta)\,d\theta$$

Building a Probability Model

(vi) Estimate the parameters (of the mixing distribution) by fitting the aggregate distribution to the observed data.

(vii) Use the model to solve the marketing decision problem/provide the required information.

Outline

Problem 1: Projecting Customer Retention Rates
(Modelling Discrete-Time Duration Data)

Problem 2: Predicting New Product Trial
(Modelling Continuous-Time Duration Data)

Problem 3: Estimating Concentration in Champagne Purchasing
(Modelling Count Data)

Problem 4: Test/Roll Decisions in Segmentation-based Direct Marketing
(Modelling “Choice” Data)

Problem 2:
Predicting New Product Trial

(Modelling Continuous-Time Duration Data)

Background

Ace Snackfoods, Inc. has developed a new shelf-stable juice product called Kiwi Bubbles. Before deciding whether or not to “go national” with the new product, the marketing manager for Kiwi Bubbles has decided to commission a year-long test market using IRI’s BehaviorScan service, with a view to getting a clearer picture of the product’s potential.

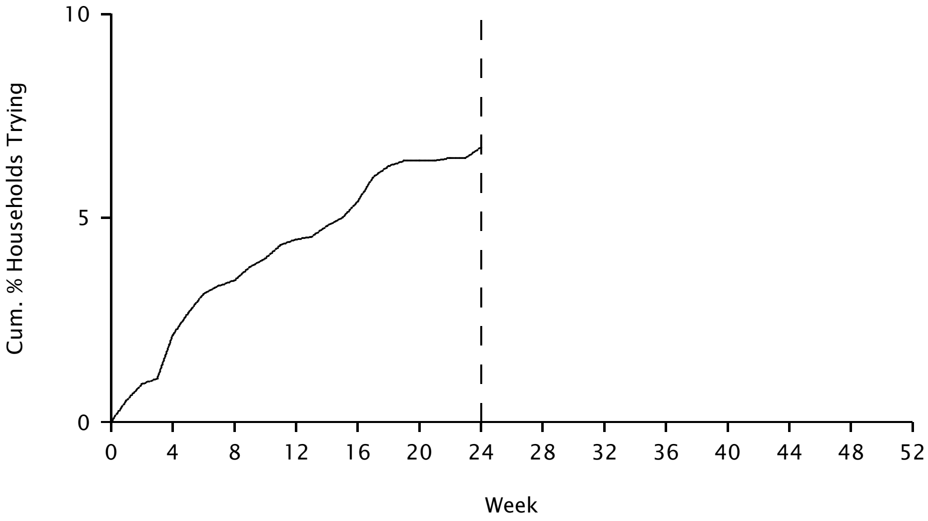
The product has now been under test for 24 weeks. On hand is a dataset documenting the number of households that have made a trial purchase by the end of each week. (The total size of the panel is 1499 households.)

The marketing manager for Kiwi Bubbles would like a forecast of the product’s year-end performance in the test market. First, she wants a forecast of the number of households that will have made a trial purchase by week 52.

Kiwi Bubbles Cumulative Trial

Week	# Households	Week	# Households
1	8	13	68
2	14	14	72
3	16	15	75
4	32	16	81
5	40	17	90
6	47	18	94
7	50	19	96
8	52	20	96
9	57	21	96
10	60	22	97
11	65	23	97
12	67	24	101

Kiwi Bubbles Cumulative Trial



Developing a Model of Trial Purchasing

- Start at the individual-level then aggregate.

  Q: What is the individual-level behavior of interest?

  A: Time (since new product launch) of trial purchase.

- We don't know exactly what is driving the behavior ⇒ treat it as a random variable.

The Individual-Level Model

- Let T denote the random variable of interest, and t denote a particular realization.
- Assume time-to-trial is characterized by the exponential distribution with parameter λ (which represents an individual's trial rate).
- The probability that an individual has tried by time t is given by:

$$F(t \mid \lambda) = P(T \le t \mid \lambda) = 1 - e^{-\lambda t} .$$

Distribution of Trial Rates

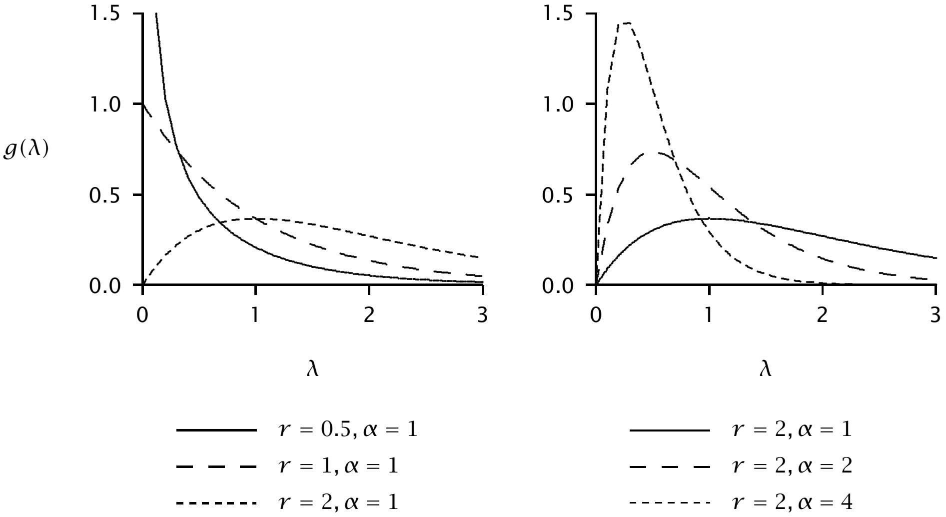
- Assume trial rates are distributed across the population according to a gamma distribution:

$$g(\lambda \mid r, \alpha) = \frac{\alpha^r \lambda^{r-1} e^{-\alpha\lambda}}{\Gamma(r)}$$

  where r is the “shape” parameter and α is the “scale” parameter.

- The gamma distribution is a flexible (unimodal) distribution . . . and is mathematically convenient.

Illustrative Gamma Density Functions



Market-Level Model

The cumulative distribution of time-to-trial at the market-level is given by:

$$\begin{aligned} P(T \le t \mid r, \alpha) &= \int_0^\infty P(T \le t|\lambda)\, g(\lambda \mid r, \alpha)\, d\lambda \\ &= 1 - \left(\frac{\alpha}{\alpha + t}\right)^r \end{aligned}$$

We call this the “exponential-gamma” model.

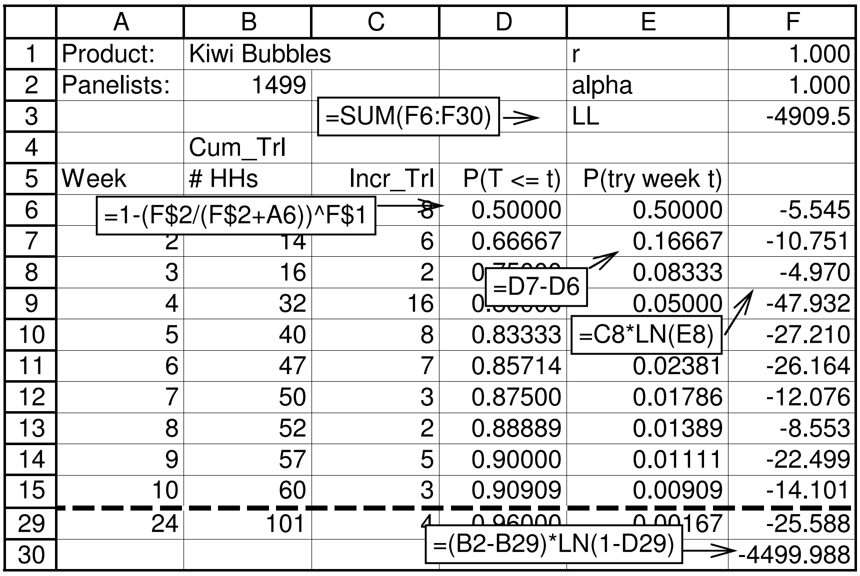
Estimating Model Parameters

The log-likelihood function is defined as:

$$\begin{aligned} LL(r, \alpha|\text{data}) = 8 \times \ln[P(0 < T \leq 1)] &\quad + \\ 6 \times \ln[P(1 < T \leq 2)] &\quad + \\ \ldots &\quad + \\ 4 \times \ln[P(23 < T \leq 24)] &+ \\ (1499 - 101) \times \ln[P(T > 24)] & \end{aligned}$$

The maximum value of the log-likelihood function is $LL = -681.4$, which occurs at $\hat{r} = 0.050$ and $\hat{\alpha} = 7.973$.

Estimating Model Parameters



	A	B	C	D	E	F
1	Product:	Kiwi Bubbles			r	1.000
2	Panelists:	1499			alpha	1.000
3					LL	-4909.5
4		Cum_Trl				
5	Week	# HHs	Incr_Trl	P(T <= t)	P(try week t)	
6			8	0.50000	0.50000	-5.545
7	2	14	6	0.66667	0.16667	-10.751
8	3	16	2	0.75000	0.08333	-4.970
9	4	32	16	0.80000	0.05000	-47.932
10	5	40	8	0.83333		-27.210
11	6	47	7	0.85714	0.02381	-26.164
12	7	50	3	0.87500	0.01786	-12.076
13	8	52	2	0.88889	0.01389	-8.553
14	9	57	5	0.90000	0.01111	-22.499
15	10	60	3	0.90909	0.00909	-14.101
29	24	101	4	0.96000	0.00167	-25.588
30						-4499.988

=SUM(F6:F30)

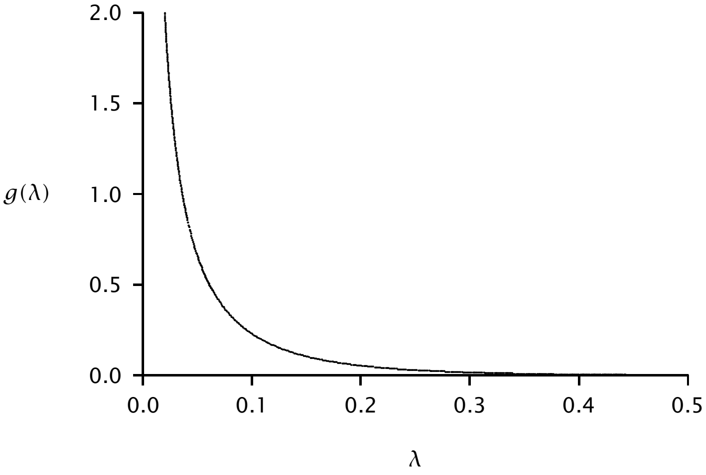
=1-(F$2/(F$2+A6))^F$1

=D7-D6

=C8*LN(E8)

=(B2-B29)*LN(1-D29)

Estimated Distribution of Λ



$\hat{r} = 0.050, \hat{\alpha} = 7.973$

Forecasting Trial

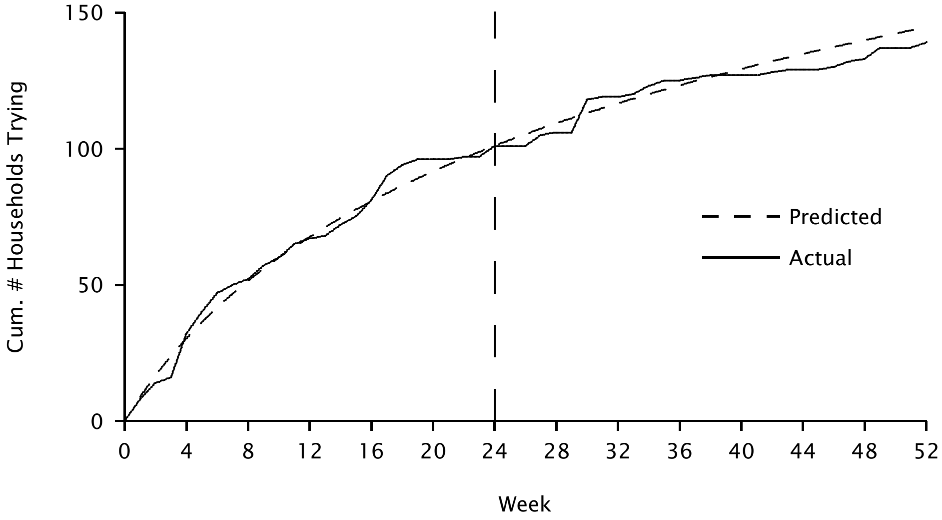
- $F(t)$ represents the probability that a randomly chosen household has made a trial purchase by time t, where $t = 0$ corresponds to the launch of the new product.
- Let $T(t)$ = cumulative # households that have made a trial purchase by time t:

$$\begin{aligned} E[T(t)] &= N \times \hat{F}(t) \\ &= N \left\{ 1 - \left(\frac{\hat{\alpha}}{\hat{\alpha} + t} \right)^{\hat{r}} \right\} . \end{aligned}$$

  where N is the panel size.

- Use projection factors for market-level estimates.

Cumulative Trial Forecast



Further Model Extensions

- Add a “never triers” parameter.
- Incorporate the effects of marketing covariates.
- Model repeat sales using a “depth of repeat” formulation, where transitions from one repeat class to the next are modeled using an “exponential-gamma”-type model.

Concepts and Tools Introduced

- Modelling continuous-time (single-event) duration data
- Models of new product trial

Further Reading

Fader, Peter S., Bruce G. S. Hardie, and Robert Zeithammer (2003), “Forecasting New Product Trial in a Controlled Test Market Environment,” *Journal of Forecasting*, **22** (August), 391–410.

Hardie, Bruce G. S., Peter S. Fader, and Michael Wisniewski (1998), “An Empirical Comparison of New Product Trial Forecasting Models,” *Journal of Forecasting*, **17** (June–July), 209–229.

Kalbfleisch, John D. and Ross L. Prentice (2002), *The Statistical Analysis of Failure Time Data*, 2nd edn, Hoboken, NJ: Wiley

Lawless, J. F. (2002), *Statistical Models and Methods for Lifetime Data*, 2nd edn, Hoboken, NJ: Wiley.

Problem 3:
Estimating Concentration in Champagne Purchasing

(Modelling Count Data)

Problem

Marketers often talk about the “80/20 rule” — 80% of sales (or revenues or profits) come from 20% of the customers.

Consider the following data on the number of bottles of champagne purchased in a year by a sample of 568 French households:

# Bottles	0	1	2	3	4	5	6	7	8+
Frequency	400	60	30	20	8	8	9	6	27

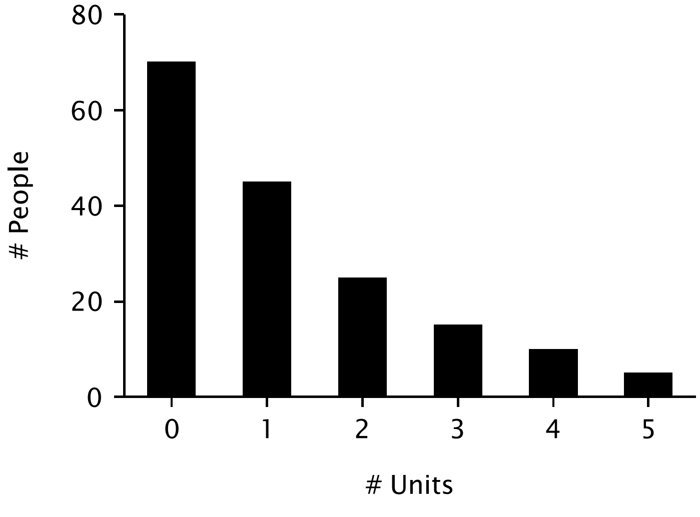
What percentage of buyers account for 80% of champagne purchasing? 50% of champagne purchasing?

Concentration 101

- Concentration in customer purchasing means that a small proportion of customers make a large proportion of the total purchases of the product.
- A *Lorenz curve* is used to illustrate the degree of inequality in the distribution of a quantity of interest (e.g., purchasing, income, wealth).
  - The Lorenz curve $L(p)$ is the proportion of total purchases accounted for by the bottom pth percentile of purchasers.
  - Constructed using the distribution of purchases.

Concentration 101

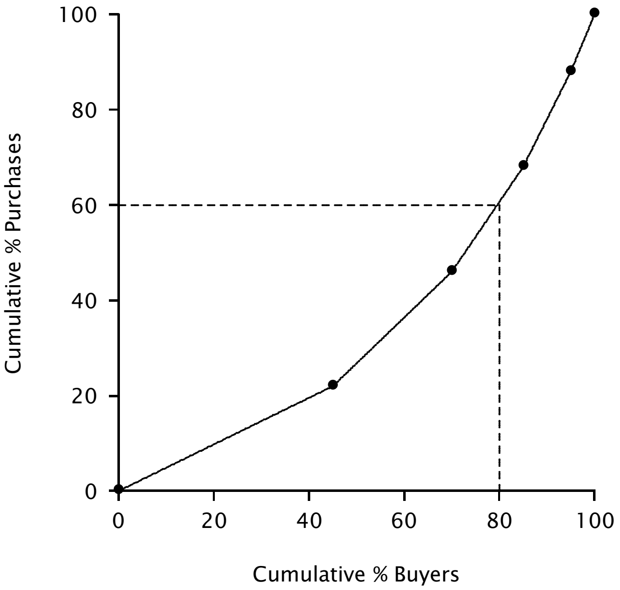
Hypothetical distribution of purchases:



Concentration 101

# Units	# People	Total Units	% Buyers	% Purchases	Cum. % Buyers	Cum. % Purchases
0	70	0	0%	0%	0%	0%
1	45	45	45%	22%	45%	22%
2	25	50	25%	24%	70%	46%
3	15	45	15%	22%	85%	68%
4	10	40	10%	20%	95%	88%
5	5	25	5%	12%	100%	100%

Lorenz Curve



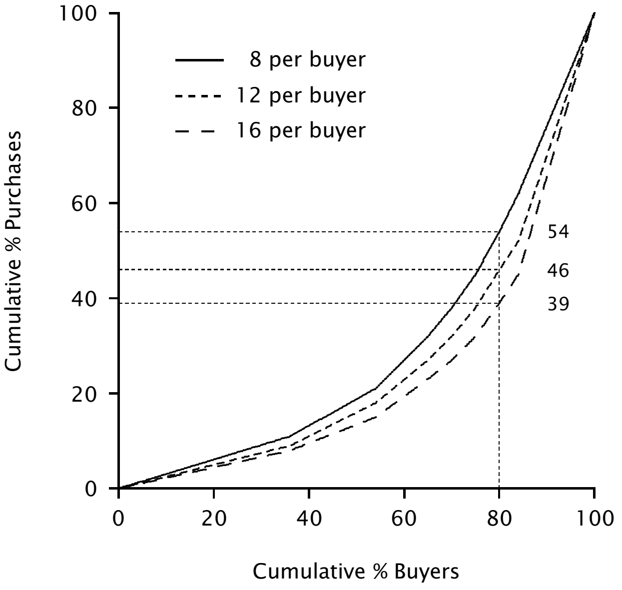
Back to the Data . . .

# Bottles	0	1	2	3	4	5	6	7	8+
Frequency	400	60	30	20	8	8	9	6	27

How many purchases occur in the 8+ cell?

- Do we assume 8 bottles per buyer? 12 per buyer? 16 per buyer?

Associated Lorenz Curves



Modelling Objective

We need to infer the full distribution from the right-censored data ... from which we can create the Lorenz curve.

⟶ Develop a model that enables us to estimate the number of people making $0, 1, 2, \ldots, 7, 8, 9, 10, \ldots$ purchases of champagne in a year.

Model Development

- Let the random variable X denote the number of bottles purchased in a year.
- At the individual-level, X is assumed to be Poisson distributed with (purchase) rate parameter λ:

$$P(X = x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

- Purchase rates (λ) are distributed across the population according to a gamma distribution:

$$g(\lambda \mid r, \alpha) = \frac{\alpha^r \lambda^{r-1} e^{-\alpha\lambda}}{\Gamma(r)}$$

Model Development

- The distribution of purchases at the population-level is given by:

$$\begin{aligned} P(X = x \mid r, \alpha) &= \int_0^\infty P(X = x|\lambda)\, g(\lambda \mid r, \alpha)\, d\lambda \\ &= \frac{\Gamma(r + x)}{\Gamma(r)x!} \left(\frac{\alpha}{\alpha + 1}\right)^r \left(\frac{1}{\alpha + 1}\right)^x \end{aligned}$$

  This is called the Negative Binomial Distribution, or NBD model.

- The mean of the NBD is given by $E(X) = r/\alpha$.

Computing NBD Probabilities

- Note that

$$\frac{P(X = x)}{P(X = x - 1)} = \frac{r + x - 1}{x(\alpha + 1)}$$

- We can therefore compute NBD probabilities using the following *forward recursion* formula:

$$P(X = x) = \begin{cases} \left(\dfrac{\alpha}{\alpha + 1}\right)^r & x = 0 \\ \dfrac{r + x - 1}{x(\alpha + 1)} \times P(X = x - 1) & x \geq 1 \end{cases}$$

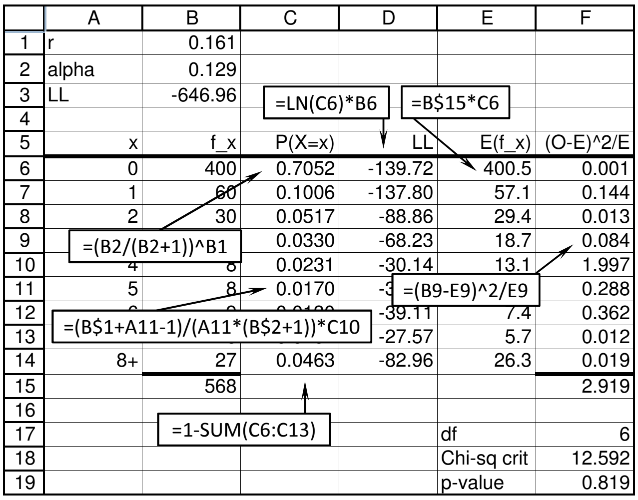
Estimating Model Parameters

The log-likelihood function is defined as:

$$\begin{aligned} LL(r, \alpha|\text{data}) = {} & 400 \times \ln[P(X=0)] + \\ & 60 \times \ln[P(X=1)] + \\ & \ldots \qquad\qquad + \\ & 6 \times \ln[P(X=7)] + \\ & 27 \times \ln[P(X \geq 8)] \end{aligned}$$

The maximum value of the log-likelihood function is $LL = -646.96$, which occurs at $\hat{r} = 0.161$ and $\hat{\alpha} = 0.129$.

Estimating Model Parameters



	A	B	C	D	E	F
1	r	0.161				
2	alpha	0.129				
3	LL	-646.96				
4						
5	x	f_x	P(X=x)	LL	E(f_x)	(O-E)^2/E
6	0	400	0.7052	-139.72	400.5	0.001
7	1	60	0.1006	-137.80	57.1	0.144
8	2	30	0.0517	-88.86	29.4	0.013
9	[illegible]	[illegible]	0.0330	-68.23	18.7	0.084
10	4	8	0.0231	-30.14	13.1	1.997
11	5	8	0.0170	[illegible]	[illegible]	0.288
12	[illegible]	[illegible]	[illegible]	-39.11	7.4	0.362
13	[illegible]	[illegible]	[illegible]	-27.57	5.7	0.012
14	8+	27	0.0463	-82.96	26.3	0.019
15		568				2.919
16						
17					df	6
18					Chi-sq crit	12.592
19					p-value	0.819

=LN(C6)*B6

=B$15*C6

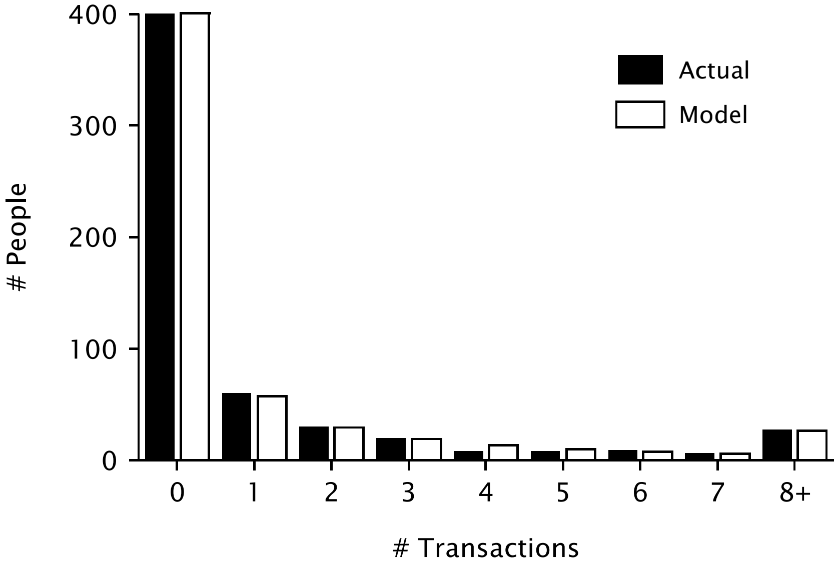
=(B2/(B2+1))^B1

=(B9-E9)^2/E9

=(B$1+A11-1)/(A11*(B$2+1))*C10

=1-SUM(C6:C13)

Model Fit



Chi-square Goodness-of-Fit Statistic

Does the distribution $F(x|\boldsymbol{\theta})$, with s model parameters denoted by $\boldsymbol{\theta}$, provide a good fit to the sample data?

- Divide the sample into k mutually exclusive and collectively exhaustive groups.
- Let f_i $(i = 1, \ldots, k)$ be the number of sample observations in group i, p_i the probability of belonging to group i, and n the sample size.

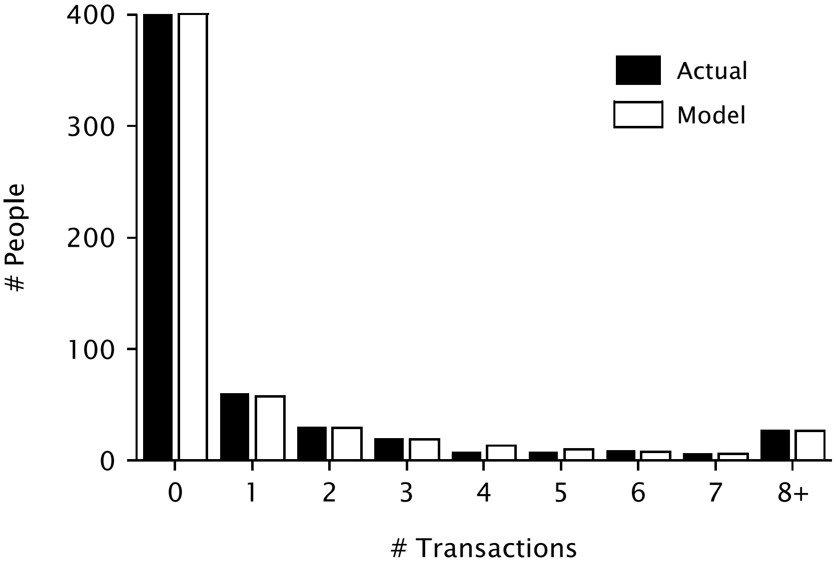
Chi-square Goodness-of-Fit Statistic

- Compute the test statistic

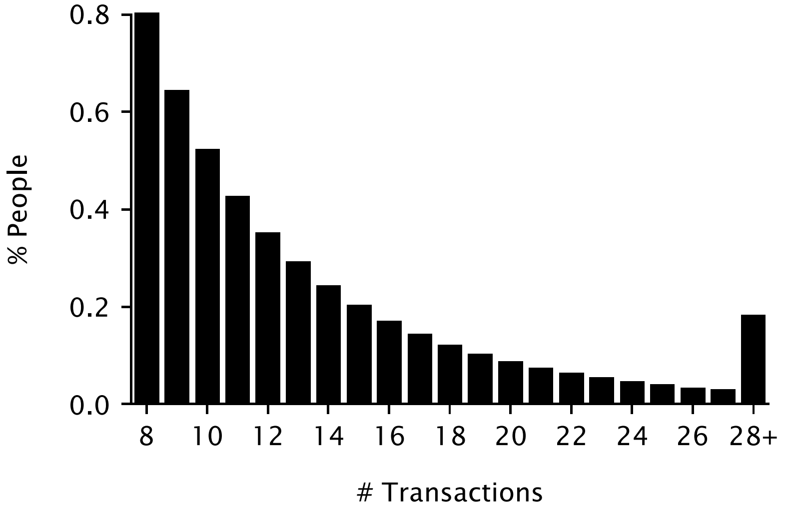
$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - np_i)^2}{np_i}$$

- Reject the null hypothesis that the observed data come from $F(x|\boldsymbol{\theta})$ if the test statistic is greater than the critical value (i.e., $\chi^2 > \chi^2_{.05,k-s-1}$).
- The critical value can be computed in Excel using the `CHIINV` function (and the corresponding p-value using the `CHIDIST` function).

Model Fit

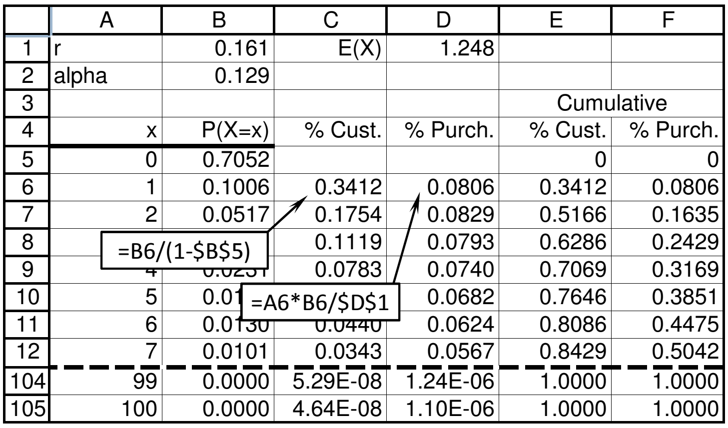


Decomposing the 8+ Cell



The mean for this group of people is 13.36 purchases per buyer … but with great variability.

Creating the Lorenz Curve

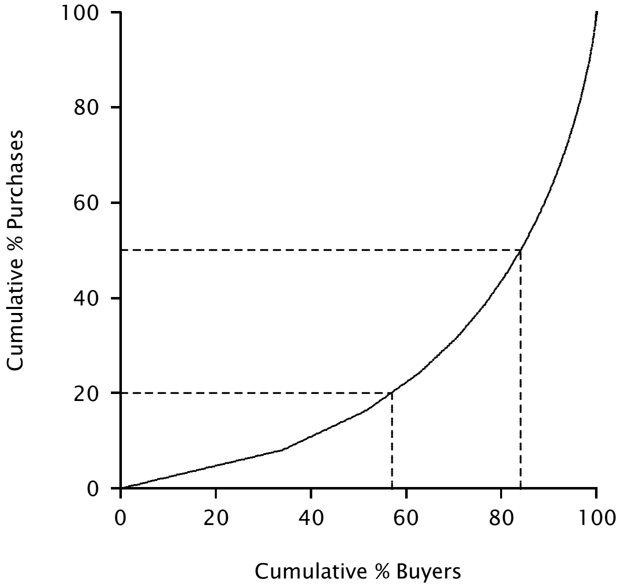


	A	B	C	D	E	F
1	r	0.161	E(X)	1.248		
2	alpha	0.129				
3					Cumulative	
4	x	P(X=x)	% Cust.	% Purch.	% Cust.	% Purch.
5	0	0.7052			0	0
6	1	0.1006	0.3412	0.0806	0.3412	0.0806
7	2	0.0517	0.1754	0.0829	0.5166	0.1635
8	[illegible]	[illegible]	0.1119	0.0793	0.6286	0.2429
9	4	0.0231	0.0783	0.0740	0.7069	0.3169
10	5	0.01	[illegible]	0.0682	0.7646	0.3851
11	6	0.0130	0.0440	0.0624	0.8086	0.4475
12	7	0.0101	0.0343	0.0567	0.8429	0.5042
104	99	0.0000	5.29E-08	1.24E-06	1.0000	1.0000
105	100	0.0000	4.64E-08	1.10E-06	1.0000	1.0000

=B6/(1-$B$5)

=A6*B6/$D$1

Lorenz Curve for Champagne Purchasing



Concepts and Tools Introduced

- Counting processes
- The NBD model
- Using models to compute concentration in customer purchasing

Further Reading

Ehrenberg, A. S. C. (1988), *Repeat-Buying*, 2nd edn, London: Charles Griffin & Company, Ltd. (Available at `http://www.empgens.com/ArticlesHome/Volume5/RepeatBuying.html`)

Greene, Jerome D. (1982), *Consumer Behavior Models for Non-Statisticians*, New York: Praeger.

Morrison, Donald G. and David C. Schmittlein (1988), "Generalizing the NBD Model for Customer Purchases: What Are the Implications and Is It Worth the Effort?" *Journal of Business and Economic Statistics*, **6** (April), 145–159.

Schmittlein, David C., Lee G. Cooper, and Donald G. Morrison (1993), "Truth in Concentration in the Land of (80/20) Laws," *Marketing Science*, **12** (Spring), 167–183.

Problem 4:

Test/Roll Decisions in Segmentation-based Direct Marketing

(Modelling "Choice" Data)

The "Segmentation" Approach

i. Divide the customer list into a set of (homogeneous) segments.

ii. Test customer response by mailing to a random sample of each segment.

iii. Rollout to segments with a response rate (RR) above some cut-off point,

$$\text{e.g., RR} > \frac{\text{cost of each mailing}}{\text{unit margin}}$$

Ben's Knick Knacks, Inc.

- A consumer durable product (unit margin = \$161.50, mailing cost per 10,000 = \$3343)
- 126 segments formed from customer database on the basis of past purchase history information
- Test mailing to 3.24% of database

Ben’s Knick Knacks, Inc.

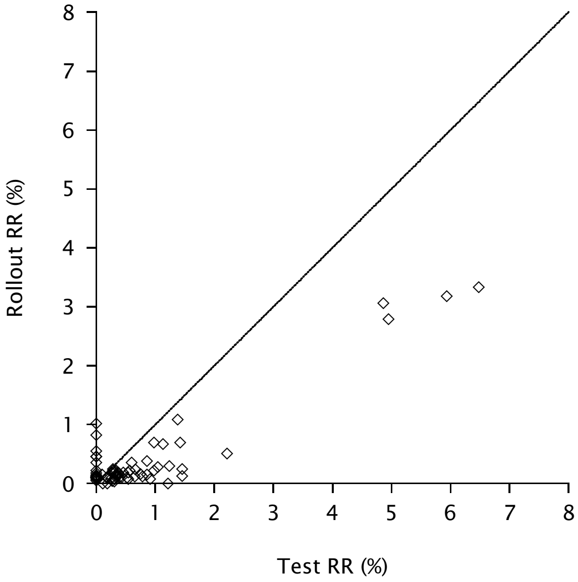
Standard approach:

- Rollout to all segments with

$$\text{Test RR} > \frac{3{,}343/10{,}000}{161.50} = 0.00207$$

- 51 segments pass this hurdle

Test vs. Actual Response Rate



Modelling Objective

Develop a model to help the manager estimate each segment's "true" response rate given the (limited) test data.

Model Development

i. Assuming all members of segment s have the same (unknown) response probability θ_s, X_s has a binomial distribution:

$$P(X_s = x_s | m_s, \theta_s) = \binom{m_s}{x_s} \theta_s^{x_s} (1 - \theta_s)^{m_s - x_s} ,$$

with $E(X_s | m_s, \theta_s) = m_s \theta_s$.

ii. Heterogeneity in θ_s is captured using a beta distribution:

$$g(\theta_s \mid \alpha, \beta) = \frac{\theta_s^{\alpha-1}(1 - \theta_s)^{\beta-1}}{B(\alpha, \beta)}$$

The Beta Binomial Model

The aggregate distribution of responses to a mailing of size m_s is given by

$$\begin{aligned} P(X_s = x_s | m_s \, \alpha, \beta) \\ &= \int_0^1 P(X_s = x_s | m_s, \theta_s) \, g(\theta_s \mid \alpha, \beta) \, d\theta_s \\ &= \binom{m_s}{x_s} \frac{B(\alpha + x_s, \beta + m_s - x_s)}{B(\alpha, \beta)} \, . \end{aligned}$$

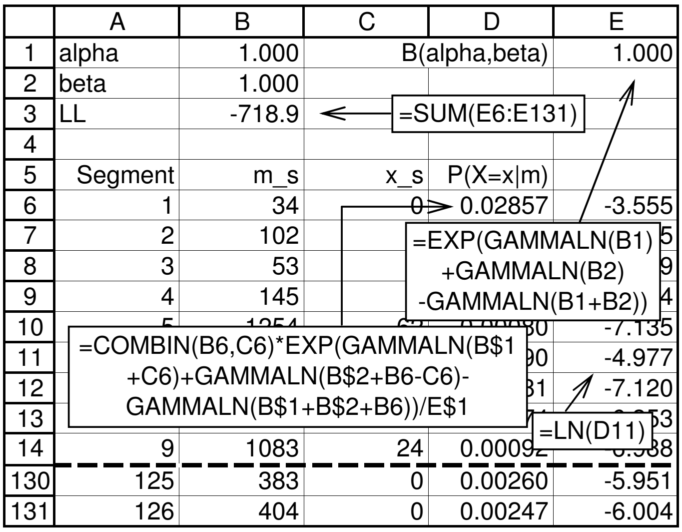
Estimating Model Parameters

The log-likelihood function is defined as:

$$\begin{aligned} LL(\alpha, \beta | \text{data}) &= \sum_{s=1}^{126} \ln[P(X_s = x_s | m_s, \alpha, \beta)] \\ &= \sum_{s=1}^{126} \ln\Bigg[\frac{m_s!}{(m_s - x_s)! \, x_s!} \underbrace{\frac{\Gamma(\alpha + x_s)\Gamma(\beta + m_s - x_s)}{\Gamma(\alpha + \beta + m_s)}}_{B(\alpha+x_s, \beta+m_s-x_s)} \underbrace{\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{1/B(\alpha,\beta)} \Bigg] \end{aligned}$$

The maximum value of the log-likelihood function is $LL = -200.5$, which occurs at $\hat{\alpha} = 0.439$ and $\hat{\beta} = 95.411$.

Estimating Model Parameters



	A	B	C	D	E
1	alpha	1.000		B(alpha,beta)	1.000
2	beta	1.000			
3	LL	-718.9			
4					
5	Segment	m_s	x_s	P(X=x|m)	
6	1	34	0	0.02857	-3.555
7	2	102	[illegible]	[illegible]	[illegible]
8	3	53	[illegible]	[illegible]	[illegible]
9	4	145	[illegible]	[illegible]	[illegible]
10	[illegible]	[illegible]	[illegible]	[illegible]	-7.135
11	[illegible]	[illegible]	[illegible]	[illegible]	-4.977
12	[illegible]	[illegible]	[illegible]	[illegible]	-7.120
13	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
14	9	1083	24	[illegible]	[illegible]
130	125	383	0	0.00260	-5.951
131	126	404	0	0.00247	-6.004

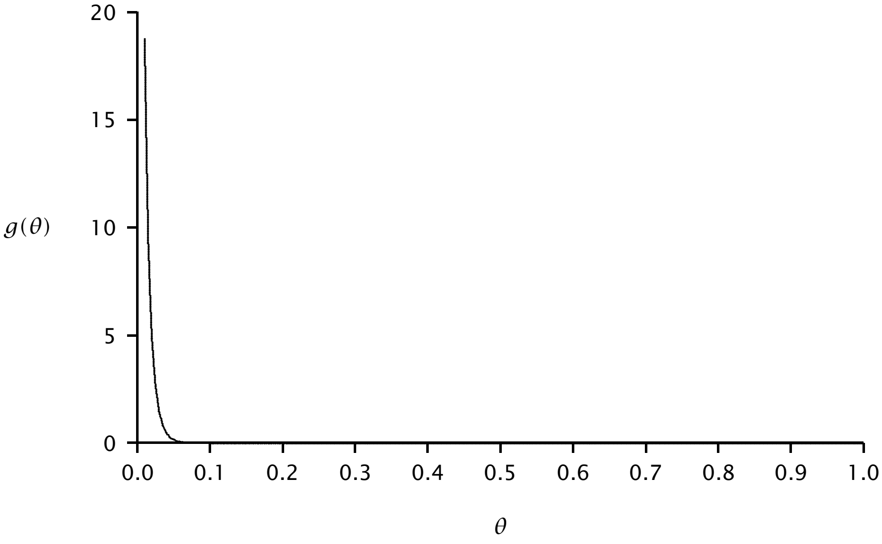
=SUM(E6:E131)

=EXP(GAMMALN(B1)+GAMMALN(B2)-GAMMALN(B1+B2))

=COMBIN(B6,C6)*EXP(GAMMALN(B$1+C6)+GAMMALN(B$2+B6-C6)-GAMMALN(B$1+B$2+B6))/E$1

=LN(D11)

Estimated Distribution of Θ



$$\hat{\alpha} = 0.439, \hat{\beta} = 95.411, \widehat{E(\Theta)} = 0.0046$$

Applying the Model

What is our best guess of θ_s given a response of x_s to a test mailing of size m_s?

Intuitively, we would expect

$$E(\Theta_s|x_s, m_s) \approx \omega\frac{\alpha}{\alpha+\beta} + (1-\omega)\frac{x_s}{m_s}$$

Bayes' Theorem

- The *prior distribution* $g(\theta)$ captures the possible values θ can take on, prior to collecting any information about the specific individual.
- The *posterior distribution* $g(\theta|x)$ is the conditional distribution of θ, given the observed data x. It represents our updated opinion about the possible values θ can take on, now that we have some information x about the specific individual.
- According to Bayes' Theorem:

$$g(\theta|x) = \frac{f(x|\theta)g(\theta)}{\int f(x|\theta)g(\theta)\,d\theta}$$

Bayes' Theorem

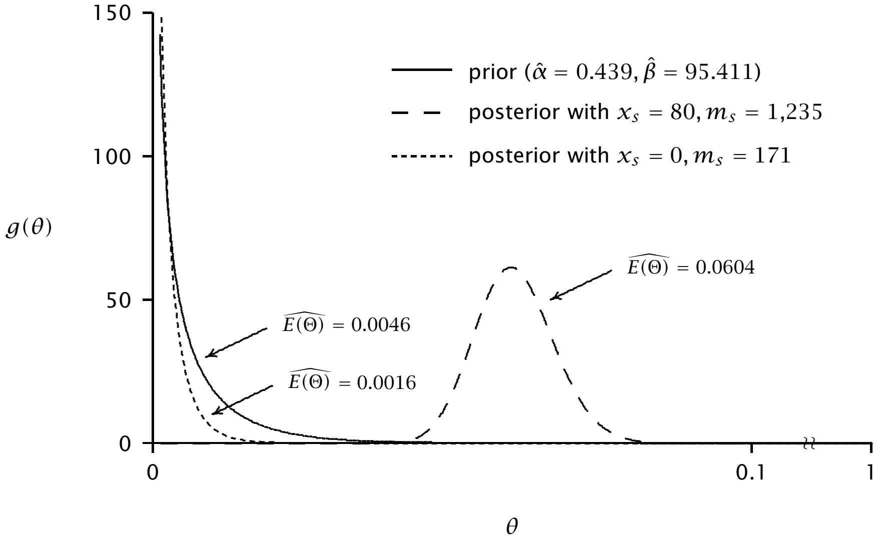
For the beta-binomial model, we have:

$$g(\theta_s|X_s = x_s, m_s) = \frac{\overbrace{P(X_s = x_s|m_s, \theta_s)}^{\text{binomial}}\,\overbrace{g(\theta_s)}^{\text{beta}}}{\underbrace{\int_0^1 P(X_s = x_s|m_s, \theta_s)\, g(\theta_s)\, d\theta_s}_{\text{beta-binomial}}}$$

$$= \frac{1}{B(\alpha + x_s, \beta + m_s - x_s)}\theta_s^{\alpha + x_s - 1}(1 - \theta_s)^{\beta + m_s - x_s - 1}$$

which is a beta distribution with parameters $\alpha + x_s$ and $\beta + m_s - x_s$.

Distribution of Θ



Applying the Model

Recall that the mean of the beta distribution is $\alpha/(\alpha+\beta)$. Therefore

$$E(\Theta_s|X_s = x_s, m_s) = \frac{\alpha + x_s}{\alpha + \beta + m_s}$$

which can be written as

$$\left(\frac{\alpha+\beta}{\alpha+\beta+m_s}\right)\frac{\alpha}{\alpha+\beta} + \left(\frac{m_s}{\alpha+\beta+m_s}\right)\frac{x_s}{m_s}$$

- a weighted average of the test RR (x_s/m_s) and the population mean ($\alpha/(\alpha+\beta)$).
- "Regressing the test RR to the mean"

Model-Based Decision Rule

- Rollout to segments with:

$$E(\Theta_s|X_s = x_s, m_s) > \frac{3{,}343/10{,}000}{161.5} = 0.00207$$

- 66 segments pass this hurdle
- To test this model, we compare model predictions with managers' actions. (We also examine the performance of the "standard" approach.)

Results

	Standard	Manager	Model
# Segments (Rule)	51		66
# Segments (Act.)	46	71	53
Contacts	682,392	858,728	732,675
Responses	4,463	4,804	4,582
Profit	$492,651	$488,773	$495,060

Use of model results in a profit increase of $6,287; 126,053 fewer contacts, saved for another offering.

Concepts and Tools Introduced

- “Choice” processes
- The Beta Binomial model
- “Regression-to-the-mean” and the use of models to capture such an effect
- Bayes’ Theorem (and “empirical Bayes” methods)
- Using “empirical Bayes” methods in the development of targeted marketing campaigns

Further Reading

Colombo, Richard and Donald G. Morrison (1988), “Blacklisting Social Science Departments with Poor Ph.D. Submission Rates,” *Management Science*, **34** (June), 696–706.

Morwitz, Vicki G. and David C. Schmittlein (1998), “Testing New Direct Marketing Offerings: The Interplay of Management Judgment and Statistical Models,” *Management Science*, **44** (May), 610–628.

Discussion

Recap

The preceding four problems introduce simple models for three behavioral processes:

- Timing ⟶ "when"
- Counting ⟶ "how many"
- "Choice" ⟶ "whether/which"

Phenomenon	Individual-level	Heterogeneity	Model
Timing (continuous)	exponential	gamma	EG (Pareto)
Timing (discrete) (or counting)	shifted-geometric	beta	sBG
Counting	Poisson	gamma	NBD
Choice	binomial	beta	BB

Further Applications: Timing Models

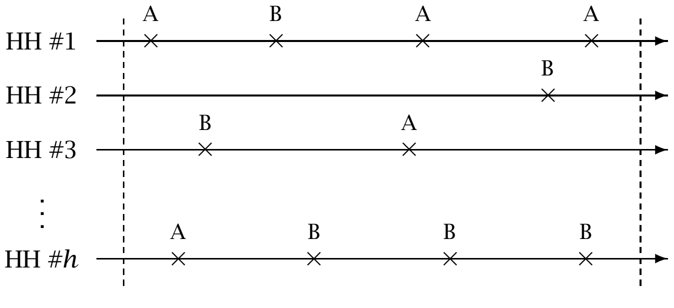
- Repeat purchasing of new products
- Response times:
  - Coupon redemptions
  - Survey response
  - Direct mail (response, returns, repeat sales)
- Other durations:
  - Salesforce job tenure
  - Length of web site browsing session
- Other positive "continuous" quantities (e.g., spend)

Further Applications: Count Models

- Repeat purchasing
- Salesforce productivity/allocation
- Number of page views during a web site browsing session
- Exposure distributions for banner ads

Further Applications: "Choice" Models

- Brand choice



- Media exposure
- Multibrand choice (BB → Dirichlet Multinomial)
- Taste tests (discrimination tests)
- "Click-through" behavior

Integrated Models

More complex behavioral phenomena can be captured by combining models from each of these processes:

- Counting + Timing
  - catalog purchases (purchasing | “alive” & “death” process)
  - “stickiness” (# visits & duration/visit)
- Counting + Counting
  - purchase volume (# transactions & units/transaction)
  - page views/month (# visits & pages/visit)
- Counting + Choice
  - brand purchasing (category purchasing & brand choice)
  - “conversion” behavior (# visits & buy/not-buy)

A Template for Integrated Models

		Stage 2		
		Counting	Timing	Choice
Stage 1	Counting			
	Timing			
	Choice			

Further Issues

Relaxing usual assumptions:

- Non-exponential purchasing (greater regularity)
  ⟶ non-Poisson counts
- Non-gamma/beta heterogeneity (e.g., “hard core” nonbuyers, “hard core” loyals)
- Nonstationarity — latent traits vary over time

The basic models are quite robust to these departures.

Extensions

- Latent class/finite mixture models
- Introducing covariate effects
- Hierarchical Bayes (HB) methods

The Excel spreadsheets associated with this tutorial, along with electronic copies of the tutorial materials, can be found at:

`http://brucehardie.com/talks.html`

An annotated list of key books for those interested in applied probability modelling can be found at:

`http://brucehardie.com/notes/001/`