Modeling the Evolution of Repeat Buying

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Abstract

The negative binomial (NBD) model, as used to characterize repeat-purchasing in numerous markets, assumes stationary individual-level buying rates. In many situations (e.g., for new products), this assumption is rather tenuous — repeat buying behavior will evolve over time, thereby requiring the use of a nonstationary model to properly capture and forecast the observed sales patterns.

We introduce a model — the nonstationary exponential-gamma (NSEG) model — that accomplishes these tasks while retaining the well-known robustness, interpretability, and other desirable properties of the basic NBD framework. Starting with an exponential-gamma process — the timing equivalent of the NBD — we introduce a stochastic renewal process, which allows consumers to revise their preferences for the new product after one or more repeat purchases of it. The nature of this renewal process can change over time, allowing for the possibility that preference revisions are common in the early stages of the new product launch, but less likely to occur after a consumer has made several repeat purchases of the product. Over time, the nonstationarity component can disappear completely, allowing the model to become exactly equivalent to the NBD for all subsequent purchases.

Our empirical analysis examines the model's validity on the basis of forecasting accuracy and parameter stability across calibration periods of different lengths. We demonstrate that NSEG performs very well on both dimensions, especially in contrast to the benchmark NBD model.

"Nothing is stationary. Everything wiggles." John Gierach, *Signs of Life.*

1 Introduction

It is common to think of mature consumer packaged goods (CPG) markets as being in equilibrium. Aside from short-term, promotion-induced fluctuations, we generally assume that patterns of repeat buying behavior are fairly stable and therefore characterize them by the NBD model (Ehrenberg 1988, Morrison and Schmittlein 1988). The NBD is a *counting* model which assumes that the number of purchases made by an individual in a given time period is characterized by the Poisson distribution with rate parameter λ, and that the distribution of these buying rates across the population follows a gamma distribution.

Turning our attention to new products, we may ask whether the repeat buying behavior of new products can also be characterized by the NBD model. There are some claims that the NBD adequately describes the repeat buying patterns of a new product (Wellan and Ehrenberg 1988), and some researchers have advocated its use for forecasting purposes within a new product context (e.g., Greene 1982) — although its forecasting performance has not been well-documented. But if we think about consumers' purchasing of a new product, we expect there will often be a period of instability during which their preferences for the new product are evolving; in other words, assuming that a consumer's latent buying rate is constant (i.e., stationary) may be rather tenuous. A simple stationary model such as the NBD may *appear* to describe the repeat buying of a new product quite well for a given (short) period of time, but if the underlying purchasing process is not stationary, model-based projections of future sales will be quite biased.

Consider the repeat (i.e., post-trial) purchasing of "Kiwi Bubbles" (a masked name for a real drink product) during a year-long test conducted in two of IRI's BehaviorScan test markets prior to its national launch. An NBD model was calibrated on the repeat sales data for the 267 panelists that tried the new product in the first 26 weeks of the test; the results are reported in Figure 1. The standard chi-square goodness-of-fit test supports the appropriateness of the NBD for this calibration dataset ($\chi^2_2 = 1.81$, $p = 0.40$) — which is what we have come to expect from the NBD.

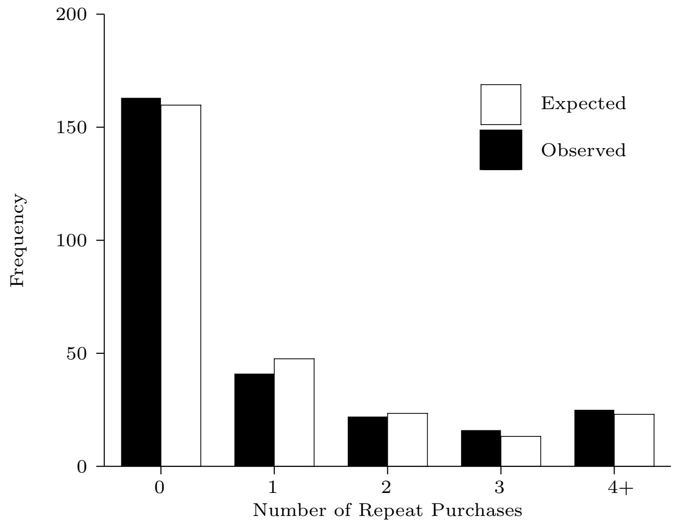

Figure 1: NBD Calibration Period Fit (Weeks 1–26)

But when we take this NBD model calibrated on the first 26 weeks and use it to predict the purchases of these 267 panelists over the next 26 weeks, we see a very different picture. Figure 2 shows the distribution of repeat purchases for weeks 27–52. Clearly the NBD-base predictions do not fit the observed purchasing behavior ($\chi^2_4 = 41.71$, $p < 0.001$). The NBD severely underpredicts the number of households with 0 repeat purchases and over-predicts the number of 4+ purchasers. Overall, the NBD over-forecasts total first-year repeat sales by 38.7%.

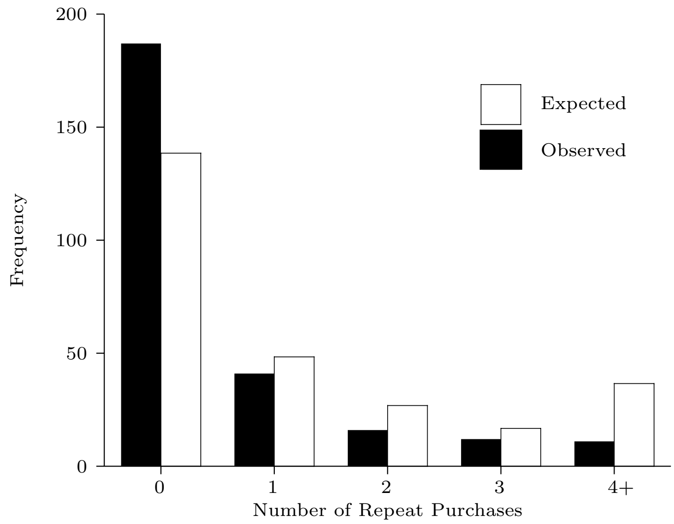

Figure 2: NBD Forecast Period Fit (Weeks 27–52)

This over-prediction indicates that the average buying rate for the first 26 weeks is higher than the average buying rate for the whole year. This is verified by comparing the NBD model parameters (r and α) obtained using the first 26 weeks of data with those obtained for the first 39 weeks and the full 52 weeks, along with the associated means (r/α). (In all cases, we focus on the repeat purchases of those 267 panelists that made a trial purchase in weeks 1–26.)

	26 wk	39 wk	52 wk
r	0.459	0.452	0.407
α	57.270	71.671	73.592
r/α	0.0080	0.0063	0.0055

It is clear that the parameter estimates are unstable; worse yet, they appear to be shifting in a systematic manner (decreasing r and increasing α) as the length of the calibration period increases. The fact that the values of the parameters seem to depend on the amount of data used to estimate them is a troubling sign about the reliability and overall validity of the NBD model in this context.

Based on these results, which are reasonably typical of many new products, we cannot assume that stationarity applies to the underlying repeat buying behavior during the early phase of a new product's life. Any model of repeat sales for a new product must account for this nonstationarity, not only when forecasting is a primary objective, but also when the purpose of the analysis is to use the model parameters to make inferences about consumer behavior. This violation of the assumption of stationarity is not just a new product phenomenon; even for established products, it is common to observe a "leakage" of repeat buyers (East and Hammond 1996).

Our goal in this paper is to develop a modeling framework that allows for nonstationary repeat buying behavior while still retaining the well-known robustness and other desirable properties of the NBD model (Morrison and Schmittlein 1988). It is easy to show that modeling repeat buying behavior using the NBD (i.e., a Poisson *counting* process with gamma heterogeneity) is equivalent to modeling it as a (stationary) exponential *timing* process with gamma heterogeneity (Gupta and Morrison 1991) — the corresponding likelihood functions differ by only a constant. In this paper we develop a nonstationary extension of this timing model, which we call NSEG (short for nonstationary exponential-gamma). We allow the degree of nonstationarity to vary over the course of a consumer's purchase history. More specifically, we assume that early repeat purchases of a new product may be associated with a relatively high likelihood of a shift in consumer preference, but as a consumer's experience with the new product increases, his behavioral pattern will often stabilize towards a stationary NBD process (although some degree of nonstationarity may still exist over the longer term).

The paper is organized as follows. In the next section we formally develop the NSEG model of repeat purchasing and explore its key properties (including its links to the basic NBD framework). The third section presents an empirical analysis in which the performance of the NSEG model is examined. We consider the two criteria discussed in the NBD example above: forecast accuracy and parameter stability (across calibration periods of different lengths). We close with a discussion of related models and future research directions.

2 Model Development

Our objective is to develop a nonstationary model of repeat (i.e., post-trial) buying for a new CPG product. The primary motivation for nonstationarity is the notion that consumers' preferences for the new product are evolving; as consumers gain more experience with the product, we expect their preferences to "settle down." As such, a desired feature of the model is that it can "evolve" to a stationary NBD-like structure as the product moves from being "new" to "established." (However, the model is not limited to new products. As it is sufficiently flexible to allow for long-term nonstationarity, it can easily be applied to established product settings in which there is nonstationarity in consumer preferences.)

Nonstationarity is modeled using a renewal process for the individual-level buying rates; a renewal can be interpreted as a revision of preferences, which may be due, perhaps, to experience with the product or some other unobservable phenomenon. A renewal can occur after any repeat purchase, but the probability of occurrence decreases as the individual has more experience with the new product (i.e., moves through higher depth-of-repeat levels). When a renewal occurs, a new value of the buying rate is drawn from the distribution of buying rates.

We now formally develop the nonstationary exponential gamma (NSEG) model of the evolution of repeat buying, which is based on the following five assumptions:

i. The probability that a consumer, who made a trial purchase at time t_0, ever makes a repeat purchase is π.

ii. Times between successive repeat purchases are exponentially distributed with parameter λ. Let the random variable T_j denote the time (since the launch of the new product) at which a consumer makes his j^{th} repeat purchase. Therefore,

$$f(t_j \,|\, t_{j-1}) = \lambda e^{-\lambda(t_j - t_{j-1})}$$

iii. Consumer-level purchase rates, λ, are distributed across the population according to a gamma distribution with shape parameter r and scale parameter α; that is

$$g(\lambda \,|\, r, \alpha) = \frac{\alpha^r \lambda^{r-1} e^{-\alpha\lambda}}{\Gamma(r)}$$

(The parameters of this distribution are assumed to be time-invariant.)

iv. Following his j^{th} repeat purchase, a consumer renews his value of λ with probability γ_j. The depth-of-repeat-specific renewal probability — constant across consumers — is driven by two parameters, ψ and θ, utilizing the following functional form

$$\gamma_j = 1 - \psi(1 - e^{-\theta j}),\ j = 1, 2, \ldots \tag{1}$$

where $\psi \in [0, 1]$ and $\theta \geq 0$.

v. Upon the occurrence of a renewal, a consumer receives a value of $\lambda = 0$ with probability ϕ. (This is equivalent to a complete rejection of the product.) With probability $1 - \phi$, the consumer draws a new value of λ, independent of his previous one, from the same gamma distribution of purchase rates that was described above.

The first assumption follows naturally from an examination of panel-based reports of new product performance, and is consistent with conventional wisdom; that is, not all triers of a new product will make a repeat purchase, as the product does not meet their expectations. The assumptions of exponential interpurchase times with gamma heterogeneity — assumptions (ii) and (iii) — are central to many modeling efforts within the marketing literature and have proved to be rather robust (Morrison and Schmittlein 1988).

The logic behind equation (1), the probability that a renewal occurs at depth-of-repeat level j, is as follows: we would expect that the probability of a consumer revising his preferences following a purchase (and consumption) experience would decrease as he gains more experience with the new product (i.e., moves to a higher depth-of-repeat level). Looking closely at equation (1), we note that as j increases, γ_j tends to $1 - \psi$. Therefore, if $\psi = 1$, the probability of a renewal tends to zero as a consumer moves to higher depth-of-repeat levels; in other words, the model evolves to a stationary process which would be consistent with the stabilization of consumer preferences. On the other hand, if $\psi < 1$, individual consumer preferences will not stabilize — which means there is long-term nonstationarity in the marketplace. (If $\theta \to \infty$, then γ_j is independent of j and γ_j equals $1 - \psi \; \forall j$.) The relationship between γ_j and j is illustrated in Figure 3 for three sets of values of ψ and θ.

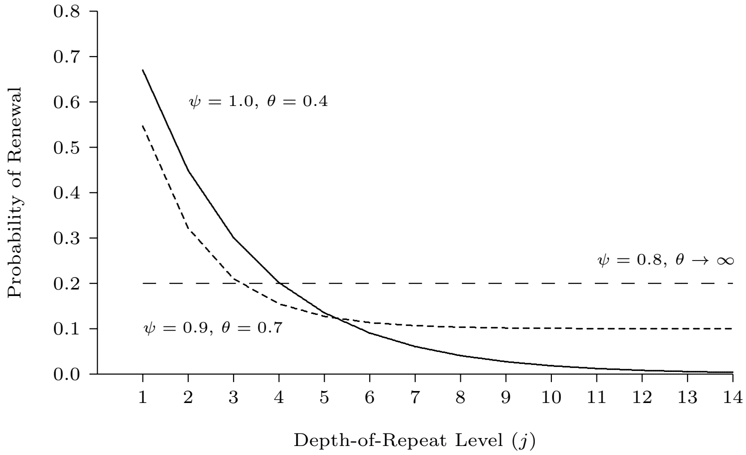

Figure 3: Probability of Renewal by Depth-of-Repeat

The fifth assumption is presented as a paramorphic, as opposed to strictly behavioral, representation of how preferences for the new product evolve. The "spike at zero" — receiving a value of $\lambda = 0$ with probability ϕ — is simply a mechanism by which consumers can "drop out" of the market for the new product even after making several repeat purchases of it; drawing a value of zero upon a renewal is viewed as being equivalent to rejecting the new product from future purchase consideration. The principle of independent renewals from a given mixing distribution was first raised in Howard's Dynamic Inference Model (Howard 1965). Similar types of renewal processes have been used by Sabavala and Morrison (1981) in their model of media exposure and Fader and Lattin (1993) in their measure of loyalty for scanner data-based choice models. However, these earlier models all utilized fixed (time-invariant) renewal processes, as opposed to the evolutionary type of renewal process that we are introducing here. (Note that equation (1) does admit a fixed (time-invariant) renewal process as a special case.)

To illustrate and convey the intuition of the proposed NSEG model, consider a consumer who makes three repeat purchases in the period $(t_0, t_c]$, where t_0 is the time of trial, t_1, t_2, t_3 are the times at which these repeat purchases occur, and t_c is the end of the model calibration period. Let us assume that if a renewal occurs (i.e., preferences are revised), it is immediately after a purchase. One behavioral "story" consistent with this is to assume that consumption immediately follows purchase, and that preference revisions would immediately follow consumption.

Given t_1, t_2, t_3, we do not know whether the consumer ever revised his preferences and, if he did, how many times and at which points in time. Let us first assume that the consumer never revised his preferences in $(t_0, t_c]$. By assumptions (i) and (ii), the conditional likelihood function for this consumer is the probability the consumer ever made a repeat purchase (π), multiplied by the product of the exponential density and survival functions, that is,

$$\begin{aligned} L(\pi, \lambda; \text{data}) &= \pi\lambda e^{-\lambda(t_1-t_0)}\lambda e^{-\lambda(t_2-t_1)}\lambda e^{-\lambda(t_3-t_2)}e^{-\lambda(t_c-t_3)} \\ &= \pi\lambda^3 e^{-\lambda(t_3-t_0)} \end{aligned}$$

Following the third assumption, the unconditional likelihood function is:

$$\begin{aligned} L(\pi, r, \alpha; \text{data}) &= \int_0^\infty L(\pi, \lambda; \text{data}) \frac{\alpha^r \lambda^{r-1} e^{-\alpha\lambda}}{\Gamma(r)}\, d\lambda \\ &= \pi \frac{\Gamma(r+3)}{\Gamma(r)} \left(\frac{1}{\alpha + t_c - t_0}\right)^3 \left(\frac{\alpha}{\alpha + t_c - t_0}\right)^r \end{aligned} \tag{2}$$

Alternatively, suppose that the consumer revised his preferences following his second repeat purchase. Let the purchasing rate λ_a reflect the consumer's preference for the new product following trial at t_0, and λ_b reflect the consumer's revised preference following his second repeat purchase. The conditional likelihood function for this consumer is therefore:

$$\begin{aligned} L(\pi, \lambda_a, \lambda_b; \text{data}) &= \pi\lambda_a e^{-\lambda_a(t_1-t_0)}\lambda_a e^{-\lambda_a(t_2-t_1)}\lambda_b e^{-\lambda_b(t_3-t_2)}e^{-\lambda_b(t_c-t_3)} \\ &= \pi\lambda_a^2 e^{-\lambda_a(t_2-t_0)}\lambda_b e^{-\lambda_b(t_c-t_2)} \end{aligned}$$

Following assumption (v), we note that the renewal resulted in a non-zero value of λ being drawn from the underlying gamma distribution, an event which occurs with probability $1 - \phi$ (i.e., the chance of avoiding rejection upon renewal). The unconditional likelihood function is therefore:

$$
\begin{aligned}
L(\pi, r, \alpha, \phi; \text{data}) &= \int_0^\infty \int_0^\infty L(\pi, \lambda_a, \lambda_b; \text{data}) \frac{\alpha^r \lambda_a^{r-1} e^{-\alpha\lambda_a}}{\Gamma(r)} (1-\phi) \frac{\alpha^r \lambda_b^{r-1} e^{-\alpha\lambda_b}}{\Gamma(r)} \, d\lambda_a \, d\lambda_b \\
&= \pi(1-\phi) \frac{\Gamma(r+2)}{\Gamma(r)} \left(\frac{1}{\alpha + t_2 - t_0}\right)^2 \left(\frac{\alpha}{\alpha + t_2 - t_0}\right)^r \\
&\quad \times \frac{\Gamma(r+1)}{\Gamma(r)} \left(\frac{1}{\alpha + t_c - t_2}\right) \left(\frac{\alpha}{\alpha + t_c - t_2}\right)^r
\end{aligned}
\tag{3}
$$

In general however, we cannot tell exactly when (or if) renewals of buying rates take place. For this consumer, the number of renewals could have ranged from zero to three. The set of eight possible renewal patterns is given in Table 1. Equation (2) is the likelihood function corresponding to the renewal pattern in row (i), and the likelihood function corresponding to the renewal pattern in row (iii) is given in equation (3). While we do not know which of the eight patterns corresponds to the consumer, we can write out the unconditional likelihood function associated with each of the possible renewal patterns and compute the consumer's overall likelihood as the weighted average of the renewal-specific likelihoods, where the weights are the probabilities of the each renewal pattern occuring.

	Renewal Occurs After Repeat Purchase			Number of Renewals
	#1	#2	#3	
(i)				0
(ii)	✓			1
(iii)		✓		1
(iv)			✓	1
(v)	✓	✓		2
(vi)	✓		✓	2
(vii)		✓	✓	2
(viii)	✓	✓	✓	3

Table 1: Feasible Renewal Patterns for Three Repeat Purchases

More generally, let $\mathbf{T}_h = \{t_1, \ldots, t_j, \ldots, t_J\}$ be the set of times at which consumer h, ($h = 1, \ldots, H$), makes his J repeat purchases in the period $(t_0, t_c]$. (Clearly $\mathbf{T}_h = \emptyset$ if $J = 0$.) When $J > 0$, the possibility of renewals occurring emerges. For a consumer making J repeat purchases in the period $(t_0, t_c]$, there are J renewal opportunities. At each renewal opportunity, a renewal either occurs or it does not; consequently, there are 2^J sets of possible renewal patterns.

Let there be $n \leq J$ renewals, and let $\mathbf{w} = \{w_i\}$ $(i = 1, \ldots, n)$ be the set of renewal points, where w_i corresponds to the depth-of-repeat level immediately following which a renewal occurs. (For the second example above, $n = 1$ and $\mathbf{w} = \{2\}$.) As we cannot tell exactly when (or if) renewals of buying rates take place, we first formulate $L(\mathbf{T}_h \,|\, \mathbf{w})$ the likelihood function conditional on a given renewal pattern, $\mathbf{w}$. (The exact form of this "conditional" likelihood function is documented in the appendix.) The probability of a given renewal pattern $\mathbf{w}$ is

$$P(\mathbf{w} \,|\, \psi, \theta) = \prod_{j \in \mathbf{w}} \gamma_j \prod_{j \in \mathbf{I} - \mathbf{w}} (1 - \gamma_j)$$

where $\mathbf{I} = \{1, 2, \ldots, J\}$. Therefore, for $J > 0$, the likelihood function associated with $\mathbf{T}_h$ is simply the weighted average of the renewal-pattern-specific likelihoods, that is,

$$L(\mathbf{T}_h) = \sum_s L(\mathbf{T}_h \,|\, \mathbf{w}_s) P(\mathbf{w}_s)$$

where the summation is over the possible renewal patterns indexed by $s = 1, 2, \ldots, 2^J$. It follows that the overall sample log-likelihood function for the NSEG model is:

$$LL = \sum_{h=1}^{H} \ln L(\mathbf{T}_h)$$

2.1 Properties of the NSEG Model

In its most general form, the NSEG model is a six-parameter model that captures nonstationarity in repeat buying behavior. It is a very flexible model that can capture many patterns of repeat buying behavior. Examples of such repeat buying phenomena include:

- *"Traditional" stationary repeat buying behavior.* If $\gamma_j = 0 \; \forall \, j$, we have a stationary process. (This is associated with $\theta \to \infty$ and $\psi = 1$.) Furthermore, if $\pi = 1$ and $\phi = 0$, we have the two parameter exponential-gamma model of stationary repeat buying behavior which is the timing counterpart of the NBD counting model (Gupta and Morrison 1991). The estimates of r and α would equal those obtained by fitting the NBD model to the data.

- *The existence of a structural "never buyers" segment.* Relaxing the above assumption that $\pi = 1$ gives us the timing equivalent of Morrison's (1969) NBD with "spike at zero" (counting) model where $1 - \pi$ is the size of the structural "never buyers" segment. This is the model proposed by Greene (1982) to capture and forecast new product repeat buying patterns.

- *The transition from a "new" to "established" product.* If $\psi = 1$ and θ is finite, then $\gamma_j \to 0$ as j increases; that is, the probability of a renewal occurring tends to zero as a consumer moves to higher depth-of-repeat levels. This means that the initial nonstationary repeat buying process evolves to a stationary process as the product becomes more established (i.e., when most buyers have made a large number of repeat purchases). Therefore the NSEG model is consistent with the notion of nonstationary buying behavior during the early stages of a new product's life and stationary buying behavior — as characterized by the NBD model — once it has become established in the marketplace.

- *Long-term nonstationarity in repeat buying.* When $\psi < 1$, the probability of renewal will *always* be non-zero which means that the repeat buying process is always nonstationary. If $\theta \to \infty$, γ_j is a constant $1 - \psi$; that is, the probability of renewal is constant across all depth-of-repeat levels. For finite θ, $\gamma_j \to 1 - \psi$ as j increases; that is, the probability of a renewal tends to the constant $1 - \psi$ as a consumer moves to higher depth-of-repeat levels. Such a model can easily capture the "leakage" of repeat buyers phenomena observed by East and Hammond (1996). In particular, if $\phi > 0$, or the underlying gamma distribution has a mode at zero ($r \leq 1$), an on-going low-level of renewals will see some consumers drawing a value of $\lambda = 0$ on a given renewal, thereby "dropping out" of the market for the product of interest.[1] Other researchers (e.g., Schmittlein, Morrison, and Colombo 1987) have proposed NBD-based models that include a "death" process. However the NSEG model is far more flexible, allowing for other forms of nonstationarity (e.g., "speeding up" and "slowing down" of latent purchase rates) beyond a simple "death" process.

[1] More precisely, for $r = 1$, the gamma density is strictly decreasing from $\lambda = 0$, while for $r < 1$, the density is strictly decreasing from an infinite peak at 0. In this case, the mode is technically undefined, but from a practical perspective is equivalent to having a mode at zero. While the gamma density is defined over $(0, \infty)$, very small values of λ, which are effectively 0, will be drawn.

2.2 Computing Repeat Sales Numbers

In order to evaluate the performance of the NSEG model or use it for forecasting purchases beyond the model calibration period, it is necessary to generate repeat purchase numbers (i.e., *counts*) from this *timing* model. Let the random variable $X_h(t)$ be the number of repeat purchases made by consumer h by time t.[2] Now

$$P(X(t) = x) = P(T_x \leq t) - P(T_{x+1} \leq t)$$

where $P(T_x \leq t)$ is the probability that the x^{th} repeat purchase occurs at or before time t. As was the case with the likelihood functions, in order to compute $P(T_x \leq t)$, we must first condition on a given renewal pattern, then uncondition by taking the weighted average over all possible renewal patterns, that is

$$P(T_x \leq t) = \sum_s P(T_x \leq t \,|\, \mathbf{w}_s) P(\mathbf{w}_s)$$

While it is straightforward to write out an expression for $P(T_x \leq t \,|\, \mathbf{w})$, computation of $P(T_x \leq t)$ quickly becomes cumbersome as x grows (since there are 2^x possible renewal patterns). We therefore propose a simulation-based approach to computing realizations of $X(t)$, from which estimates of the distribution of repeat purchases can be derived.

Simulating sales under the NSEG model is a simple exercise. For a given consumer, the simulation starts by drawing a uniform random variate to determine whether he will ever make a repeat purchase (with probability π). If this is the case, a value of λ is drawn from the gamma distribution. Using this value of λ, an exponential interpurchase time is simulated. Adding this to t_0 gives us the consumer's simulated value of t_1, the time of his first repeat purchase. If $t_1 > t$, the consumer is deemed to have made zero repeat purchases by time t (i.e., $X_h(t) = 0$) and the procedure moves on to the next consumer. If $t_1 < t$, a uniform random number is drawn to determine whether the consumer retains his value of λ (with probability $1 - \gamma_1$) or whether a renewal occurs (with probability γ_1), in which case a new value of λ is assigned. Another uniform

[2]For notational simplicity, we suppress the time origin of the counting process which is most typically t_0, the time at which the consumer made his trial purchase. Furthermore, the consumer index will also be suppressed when it is convenient to do so.

random number is drawn in the process of determining the new value of λ. With probability ϕ, a value of $\lambda = 0$ is drawn and the consumer is deemed to have rejected the new product; the consumer is recorded as having made one repeat purchase (i.e., $X_h(t) = 1$), and the procedure moves on to the next consumer. If the new value of λ is drawn from the gamma distribution (with probability $1 - \phi$), or no renewal has occurred, another exponential interpurchase time is simulated and added to t_1 to give us the consumer's simulated value of t_2, the time of his second repeat purchase. If $t_2 > t$, the procedure moves on to the next consumer; otherwise the whole process continues for this consumer until $t_j > t$ or a value of $\lambda = 0$ is drawn when a renewal occurs, at which time the number of repeat purchases made by the consumer is recorded and the procedure moves to the next consumer.

Having simulated the repeat-buying behavior of all the consumers in the sample, the empirical distribution of $X(t)$ can easily be derived. Alternatively, an estimate of the (cumulative) repeat sales volume up to time t, $R(t)$, can be obtained by summing the simulated $X_h(t)$ across the consumers, i.e., $R(t) = \sum_{h=1}^{H} X_h(t)$. A more accurate estimate of $R(t)$ and/or the empirical distribution of $X(t)$ is obtaining by repeating the simulation, say 1000 times, and taking an average of the run-specific numbers.

3 Empirical Analysis

Based on the issues raised at the start of the paper, we have two primary objectives in analyzing the performance of the proposed model. First, we want to generate accurate forecasts of future repeat purchasing behavior; second, we want to obtain a set of parameter estimates that does not vary substantially as we change the length of the calibration period. (As a reminder, the basic NBD model fared very poorly on both dimensions.) Each of these performance criteria is interesting and managerially relevant by itself, and in conjunction they provide a rigorous assessment of a model's overall validity.

We return to our "Kiwi Bubbles" dataset for a more detailed investigation of the NSEG model and some of its nested variants (including the NBD). This product is a shelf-stable juice drink, aimed primarily at children, and is sold as a multipack, with several single-serve containers bundled together. We use IRI BehaviorScan panel data, drawn from 2799 panelists in two markets. All of the models discussed in this section are estimated using the 267 panelists that tried the new product by the end of week 26.

The estimated parameters are shown in Table 2, and we briefly review their definitions and interpretations. The first column reports the estimate of π, which can be interpreted as the fraction of triers who will eventually make at least one repeat purchase. For instance, according to the full NSEG model (shown in the top row of Table 2), 48.9% of triers will eventually make one or more repeat purchases. The next two columns show the values of the shape and scale parameters (r and α) for the underlying gamma distribution that characterizes the heterogeneous purchasing rates across the repeat purchasing panelists. If/when a given panelist does complete his first repeat purchase, we can then use equation (1) to determine that there is a 57.0% chance $(1 - \psi(1 - e^{-\theta j}))$ that he will undergo a renewal immediately after this purchase. If a renewal occurs, there is a 46.5% chance (ϕ) that this panelist will drop out of the market for the new product, versus a 53.5% $(1 - \phi)$ chance that he will stay in the market and draw a new purchase rate parameter from the original gamma distribution. After a second repeat purchase, the renewal probability drops down to 36.6%, and by the sixth repeat, this probability is within 1.2% of its long-run asymptotic value of $1 - \psi = 18\%$.

	Model	π	r	α	ψ	θ	ϕ	Params	LL	BIC
(i)	Full NSEG	0.489	1.421	48.744	0.820	0.743	0.465	6	–1569.29	3172.1
(ii)	No ϕ	0.732	0.425	21.334	0.817	0.648	0	5	–1570.99	3169.9
(iii)	No ψ	0.476	1.591	51.855	1	0.294	0.345	5	–1570.80	3169.5
(iv)	No π	1	0.263	18.194	0.802	1.146	0.000	5	–1571.99	3171.9
(v)	No θ	0.523	1.128	43.521	0.754	∞	0.734	5	–1571.79	3171.5
(vi)	No ψ, ϕ	0.661	0.514	22.588	1	0.285	0	4	–1572.47	3167.3
(vii)	No π, ϕ	1	0.263	18.194	0.802	1.146	0	4	–1571.99	3166.3
(viii)	No θ, ϕ	0.931	0.288	19.415	0.720	∞	0	4	–1573.19	3168.7
(ix)	No π, θ	1	0.265	19.329	0.748	∞	0.140	4	–1573.16	3168.7
(x)	No π, ψ	1	0.280	17.805	1	0.422	0.000	4	–1576.57	3175.5
(xi)	Only ψ	1	0.261	18.878	0.731	∞	0	3	–1573.22	3163.2
(xii)	Only θ	1	0.280	17.805	1	0.422	0	3	–1576.57	3169.9
(xiii)	Only π	0.900	0.554	62.144	1	∞	0	3	–1592.12	3201.0
(xiv)	"NBD"	1	0.459	57.270	1	∞	0	2	–1592.16	3195.5

Table 2: Parameter Estimates

As we move down the table, we present nested versions of the complete NSEG model, with an increasing number of parameter constraints. For instance, for each entry in the first group of nested models, we constrain one of the model parameters by setting it to its logical "null" value. The second group constrains two parameters, and so on. At the bottom of the table we see the "NBD" model, which is merely an NSEG model with no opportunity for drop outs or renewals (i.e., a stationary exponential-gamma model).[3] Even when accounting for its four fewer parameters, this model fits significantly worse than model (i) — on the basis of both BIC and the likelihood ratio test — once again indicating the need for some type of nonstationary model component(s).

In comparing the fit of the 12 models in between Full NSEG and NBD on the basis of BIC, there is one specification, model (xi), which stands out from the others. With only three parameters, this model offers a simple, plausible behavioral story. With $\theta \to \infty$, the renewal probabilities are constant (at $1-\psi = 0.269$) over time — after any repeat purchase, the panelist will draw a new purchase rate parameter about a quarter of the time, and this nonstationary tendency never vanishes. Since $\pi = 1$ and $\phi = 0$ there is no structural drop out operating. However, since $r < 1$, the gamma density is decreasing from an effective mode at 0. This implies that a number of panelists will draw *very* small values of λ when a renewal occurs. For all intents and purposes, this is equivalent to drawing a value of $\lambda = 0$ and therefore dropping out of the market for the new product. Consequently, this model will predict some "virtual" consumer drop out (i.e., rejection of the new product after several repeat purchases), even though $\phi = 0$. The fit of this model is evident from Figure 4, which compares the model-based estimates of repeat purchasing with those observed in the calibration period. (The expected NSEG counts are generated using 1000 iterations of the simulation procedure described in section 2.2 above.) The standard chi-square goodness-of-fit test supports the appropriateness of this 3 parameter NSEG model formulation for this calibration dataset ($\chi^2_1 = 0.67$, $p = 0.41$).

An interesting way to appreciate the contrast between this three-parameter NSEG model and the NBD model is to examine the mean buying rate from the underlying gamma distribution (r/α) for each model. Notice that this ratio is considerably higher for NSEG than for NBD (0.014 vs. 0.008). This allows for a faster average buying rate at the beginning of the repeat-

[3] The estimates of r and α are exactly the same as those obtained by fitting a true NBD model. The only thing that differs is the value of LL due to the added constants in the NBD model's log-likelihood function (Gupta and Morrison 1991). Subsequent references to the NBD model are to its timing equivalent.

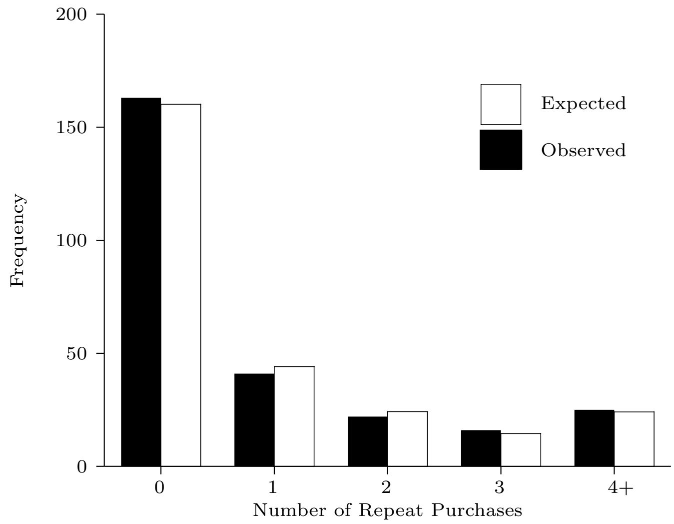

Figure 4: NSEG Calibration Period Fit (Weeks 1–26)

purchase process. However, heavy buyers (i.e., those with a high λ) are more likely to have made a renewal by any given point in time, and when a renewal occurs, it is more likely that the value of λ drawn will be less than the mean — recall that $r < 1$. This will pull down the realized average buying rate over time, allowing the histogram of repeat purchases to "slide" to the left, as evident when comparing Figures 1 and 2.

For this reason, it should come as no surprise that this 26-week NSEG model produces a far more accurate forecast of purchasing in weeks 27–52 than the NBD model for this dataset. Using 1000 iterations of the above simulation procedure, we generate the histogram of repeat purchasing in weeks 27–52 by the 267 week 1–26 triers. Figure 5 compares this estimate with the observed distribution. The good fit is quite apparent ($\chi^2_4 = 3.43$, $p = 0.49$) and stands in stark contrast to that observed for the NBD model (Figure 2.) Furthermore, the NSEG model's overall forecast of 52-week cumulative repeat-sales falls within 1% of the actual level. (Compare this to the 38.7% over-prediction associated with the NBD model.) This is dramatic evidence of the NSEG model's ability to capture and accommodate nonstationary behavior — and this is accomplished with just one additional parameter in the specification used here.

While the NSEG forecast performance is very encouraging, it does not necessarily prove that the model is properly tracking the dynamic process that is taking place as the Kiwi Bubbles product moves from introduction towards maturity. As discussed earlier in the case of the NBD

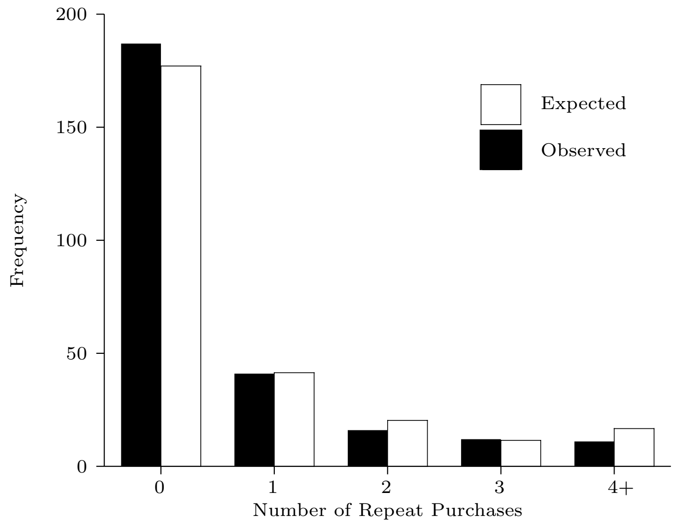

Figure 5: NSEG Forecast Period Fit (Weeks 27–52)

model, a critical aspect of a longitudinal sales model is the stability shown in its parameter estimates as the length of the calibration changes. Accordingly, we re-estimate the three-parameter NSEG model using all 52 weeks of available data for the 267 panelists who made a trial purchase by the end of week 26, and compare the resulting model fits and parameter estimates. These parameters are reported in Table 3, which also includes the equivalent numbers for the NBD model.

Parameter	NSEG		"NBD"	
	26 wk	52 wk	26 wk	52 wk
π	1	1	1	1
r	0.261	0.247	0.459	0.407
α	18.878	18.859	57.270	73.592
ψ	0.731	0.758	1	1
θ	∞	∞	∞	∞
ϕ	0	0	0	0
LL (calib)	–1573.33	–2666.63	–1592.16	–2713.12
LL (52 wk)	–2666.97	–2666.63	–2719.73	–2713.12

Table 3: Comparison of 26 and 52 Week Models

The results are quite impressive, as the NSEG parameter estimates barely change from one length of calibration period to the other, in sharp contrast to those of the NBD model. The constancy of the NSEG estimates is nicely illustrated by the log-likelihood numbers shown in the bottom row of the table. When the 26-week parameter estimates are applied to all 52 weeks of data, the resulting log-likelihood is a meager 0.34 points worse than the "optimal" model fit for the full period. Thus, despite the fact that nearly 40% of the repeat purchases in this dataset take place over the final 26 weeks, there is virtually no new information provided by them, beyond what the NSEG model is already able to glean from the repeat purchases made over the first 26 weeks. In other words, there is good reason to believe that the nonstationarity present throughout this dataset conforms well to the set of assumptions underlying the NSEG model of repeat purchasing.

4 Discussion

In this paper, we have extended the basic "NBD" framework for the modeling of repeat buying by relaxing the assumption of constant (stationary) individual-level buying rates. This is achieved by introducing a stochastic renewal process which allows consumers to revise their preferences for the product after a purchase has occurred. The resulting nonstationary exponential-gamma (NSEG) model allows for the evolution of preferences for the product while retaining the well-known robustness, interpretability, and other desirable properties of the basic NBD framework. This is especially attractive for the modeling of repeat buying for a new product in that the nature of the renewal process can change over time, allowing for the possibility that preference revisions are common in the early stages of the new product launch, but less likely to occur after a consumer has made several repeat purchases of the product. Over time, the nonstationarity component can disappear completely, allowing the model to become exactly equivalent to the NBD for all subsequent purchases.

The NSEG model was evaluated on two criteria — forecast accuracy and parameter stability — and we found that it performs quite well on both dimensions. Along the way, we examined several different specifications of the NSEG model, from the full version with six parameters down to the two-parameter model that is exactly equivalent to the ordinary NBD. Based on parsimony and model fit, we chose to focus on a three-parameter specification — one that allows for a constant 27% chance of renewing the value of λ after each repeat purchase. We estimated the models separately with 26 weeks and 52 weeks of calibration data and found virtually no differences in the parameter estimates or the quality of the model fit – something *not* observed with the (stationary) NBD model. We view this as strong evidence that the model is successfully able to capture the dynamic purchase process that is occurring throughout the data. The empirical analysis established two important concepts: (1) the repeat sales of a new product arrive from an inherently nonstationary process, and (2) conventional (stationary) modeling procedures are seriously inadequate in their ability to handle such data.

For the purpose of forecasting the repeat sales of a new product, a natural competing model would be one based on the work of Eskin (1973) and Kalwani and Silk (1980). This would involve an exponential-gamma timing process to model the time between repeat purchases, coupled with a drop-out mechanism that generates a decreasing probability of new product rejection as a household moves to higher depth-of-repeat levels. From a purely forecasting perspective, this model is robust and flexible; however it produces very misleading insights into the underlying buying behavior of the market (Fader and Hardie 1999). For example, when fitted to completely stationary (simulated) purchasing data, the estimated model parameters incorrectly tell a story of consumers dropping out (i.e., rejecting the new product) over time. Furthermore, the parameter estimates are quite unstable as the length of calibration period changes. We can therefore conclude that the Eskin/Kalwani and Silk framework is dominated by the NSEG model developed in this paper.

Other researchers have sought to relax the stationarity assumption of the NBD model — be it the pure counting form (Lenk, Rao, and Tibrewala 1993) or its timing counterpart (Gupta 1991). Both efforts use a nonhomogeneous Poisson process with an time-dependent individual-level purchase rate of the form $\lambda(t) = \lambda\psi(t)$. In other words, each individual's underlying purchase rate is multiplied by a common function of time-dependent variables; $\psi(t)$ is assumed to be a function of marketing mix variables. Cross-sectional heterogeneity is captured by assuming that λ is gamma distributed. Reflecting on this formulation, we see that it assumes that underlying consumer preferences (λ) *do not change over time* but are merely adjusted on a temporary basis by marketing mix effects. In contrast, the NSEG model represents a major change to the "NBD" framework by allowing underlying consumer preferences to change over time.

The NSEG model makes no attempt to identify the sources of nonstationarity in consumer buying rates. It is quite possible that some of the nonstationarity is due to the short-term effects of marketing mix activities. A natural extension to the NSEG model would be to incorporate marketing mix covariate effects. Gupta (1991) clearly documents how to incorporate marketing mix covariates into the basic (stationary) exponential-gamma model of repeat buying. Using the same basic approach to formulating a conditional likelihood which incorporates covariate effects, it would be a relatively straightforward exercise to add covariate effects to the NSEG model. In comparing such a model to the NSEG model, we would not be surprised to see some of the nonstationarity disappear; however, we would not expect it to completely disappear (i.e., $\hat{\gamma}_j = 0 \ \forall j$), especially in a new product setting such as that examined in the above empirical analysis.

It should be stressed that the lack of covariate effects in the basic NSEG model is by no means a shortcoming. The basic NBD model is known to work well, in spite of ignoring covariate effects. Furthermore in many situations when the NBD can be applied — both within and outside of marketing — time-varying covariates are either unimportant for the phenomena under investigation or are simply unavailable (especially in forecasting applications). The NSEG model is a natural candidate model for application in such settings.

Finally, there may be some opportunities to apply the NSEG model in other situations (besides new products) in which the NBD model would ordinarily be invoked. For instance, it might applicable to model the evolution of an entire product category, perhaps accompanied by a separate brand choice model to capture the shifts in market shares as the category develops. In some sense, any time a modeling situation calls for the NBD, the NSEG generalization might be a worthy alternative to consider.

Appendix

The exact nature of the likelihood function for consumer h depends on whether $J = 0$ or $J > 0$. If no repeat purchase is observed (i.e., $J = 0$), this is due to either (i) the consumer not liking the new product and therefore never making a repeat purchase, or (ii) the consumer not yet having the opportunity or need to make a repeat purchase. Therefore, the likelihood function for a consumer making no repeat purchases is written as:

$$L(\mathbf{T}_h) = (1-\pi) + \pi \left(\frac{\alpha}{\alpha + t_c - t_0} \right)^r$$

When $J > 0$, the possibility of renewals occurring emerges. The exact form of this "conditional" likelihood function depends on when the last renewal occurs. For the case of no renewals ($n = 0$), we have

$$L(\mathbf{T}_h \,|\, \mathbf{w}) = \pi \frac{\Gamma(r+J)}{\Gamma(r)} \left(\frac{1}{\alpha + t_c - t_0} \right)^J \left(\frac{\alpha}{\alpha + t_c - t_0} \right)^r$$

(Multiplying this by the constant $(t_c - t_0)^J / J!$ gives us the probability of J purchases in the period $(t_0, t_c]$ as computed by the NBD with "spike at zero" model, thus illustrating the equivalence of exponential-gamma timing and NBD counting processes in a stationary environment.)

For $n > 0$ renewals, with the last renewal occurring immediately following the last repeat purchase (i.e., $w_n = J$), we have

$$L(\mathbf{T}_h \,|\, \mathbf{w}) = \pi(1-\phi)^{n-1} \prod_{i=1}^{n} \left\{ \frac{\Gamma(r + w_i - w_{i-1})}{\Gamma(r)} \left(\frac{1}{\alpha + t_{w_i} - t_{w_{i-1}}} \right)^{w_i - w_{i-1}} \left(\frac{\alpha}{\alpha + t_{w_i} - t_{w_{i-1}}} \right)^r \right\}$$
$$\times \left\{ \phi + (1-\phi) \left(\frac{\alpha}{\alpha + t_c - t_{w_n}} \right)^r \right\}$$

where $w_0 = 0$. Alternatively, if the final renewal occurs some time *before* the last repeat purchase (i.e., $w_n < J$), we have

$$L(\mathbf{T}_h \,|\, \mathbf{w}) = \pi(1-\phi)^{n} \prod_{i=1}^{n} \left\{ \frac{\Gamma(r + w_i - w_{i-1})}{\Gamma(r)} \left(\frac{1}{\alpha + t_{w_i} - t_{w_{i-1}}} \right)^{w_i - w_{i-1}} \left(\frac{\alpha}{\alpha + t_{w_i} - t_{w_{i-1}}} \right)^r \right\}$$
$$\times \left\{ \frac{\Gamma(r + J - w_n)}{\Gamma(r)} \left(\frac{1}{\alpha + t_c - t_{w_n}} \right)^{J - w_n} \left(\frac{\alpha}{\alpha + t_c - t_{w_n}} \right)^r \right\}$$

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