Additional Results for the Pareto/NBD Model

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Abstract

This note derives expressions for i) the raw moments of the posterior distribution of Λ and M, ii) the marginal posterior distributions of Λ and M, and iii) $P(\text{alive at } T+t \,|\, x, t_x, T)$.

1 Preliminaries

Recall the basic Pareto/NBD model results (Fader and Hardie 2005):

i) The individual-level likelihood function for someone with purchase history (x, t_x, T) is

$$L(\lambda, \mu \,|\, x, t_x, T) = \frac{\lambda^x \mu}{\lambda+\mu} e^{-(\lambda+\mu)t_x} + \frac{\lambda^{x+1}}{\lambda+\mu} e^{-(\lambda+\mu)T} . \tag{1}$$

ii) The likelihood function for a randomly chosen individual with purchase history (x, t_x, T) is

$$L(r, \alpha, s, \beta \,|\, x, t_x, T) = \frac{\Gamma(r+x)\alpha^r \beta^s}{\Gamma(r)} \left\{ \left(\frac{s}{r+s+x}\right) \mathsf{A}_1 + \left(\frac{r+x}{r+s+x}\right) \mathsf{A}_2 \right\} \tag{2}$$

where

$$\mathsf{A}_1 = \begin{cases} \dfrac{{}_2F_1\big(r+s+x, s+1; r+s+x+1; \frac{\alpha-\beta}{\alpha+t_x}\big)}{(\alpha+t_x)^{r+s+x}} & \text{if } \alpha \geq \beta \\[2ex] \dfrac{{}_2F_1\big(r+s+x, r+x; r+s+x+1; \frac{\beta-\alpha}{\beta+t_x}\big)}{(\beta+t_x)^{r+s+x}} & \text{if } \alpha < \beta \end{cases}$$

and

$$\mathsf{A}_2 = \begin{cases} \dfrac{{}_2F_1\big(r+s+x, s; r+s+x+1; \frac{\alpha-\beta}{\alpha+T}\big)}{(\alpha+T)^{r+s+x}} & \text{if } \alpha \geq \beta \\[2ex] \dfrac{{}_2F_1\big(r+s+x, r+x+1; r+s+x+1; \frac{\beta-\alpha}{\beta+T}\big)}{(\beta+T)^{r+s+x}} & \text{if } \alpha < \beta \end{cases}$$

iii) The probability that a customer with purchase history (x, t_x, T) is "alive" at time T is the probability that the (unobserved) time at which he "dies" (ω) occurs after T, which is given by

$$P(\Omega > T \,|\, \lambda, \mu; x, t_x, T) = \frac{\lambda^x e^{-(\lambda+\mu)T}}{L(\lambda, \mu \,|\, x, t_x, T)} . \tag{3}$$

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iv) The joint posterior distribution of Λ and M is

$$g(\lambda, \mu \,|\, r, \alpha, s, \beta; x, t_x, T) = \frac{L(\lambda, \mu \,|\, x, t_x, T) g(\lambda \,|\, r, \alpha) g(\mu \,|\, s, \beta)}{L(r, \alpha, s, \beta \,|\, x, t_x, T)}\,, \tag{4}$$

where

$$g(\lambda \,|\, r, \alpha) = \frac{\alpha^r \lambda^{r-1} e^{-\lambda\alpha}}{\Gamma(r)} \tag{5}$$

is the gamma distribution that captures heterogeneity in transaction rates across customers, and

$$g(\mu \,|\, s, \beta) = \frac{\beta^s \mu^{s-1} e^{-\mu\beta}}{\Gamma(s)} \tag{6}$$

is the gamma distribution that captures heterogeneity in "death" rates across customers.

One function we will come across is the confluent hypergeometric function of the second kind (also known as the Tricomi function),[1] which has the following integral representation:

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{c-a-1} dt\,.$$

A simple generalization of this is (Gradshteyn and Ryzhik 1994, equation 3.383–5)

$$\int_0^\infty e^{-px} x^{q-1} (1+ax)^{-v} dx = a^{-q} \Gamma(q) \Psi(q, q+1-v; p/a)\,. \tag{7}$$

2 Raw Moments of the Posterior Distribution of Λ and M

For $l, m = 0, 1, 2, \ldots$, $\mathbb{E}[\Lambda^l M^m \,|\, r, \alpha, s, \beta; x, t_x, T]$ is the (l, m)th raw moment of the joint posterior distribution of Λ and M. We derive an expression for this quantity in the following manner.

$$\begin{aligned}
&\mathbb{E}[\Lambda^l M^m \,|\, r, \alpha, s, \beta; x, t_x, T] \\
&\quad = \int_0^\infty \int_0^\infty \lambda^l \mu^m g(\lambda, \mu \,|\, r, \alpha, s, \beta; x, t_x, T)\, d\lambda\, d\mu \\
&\quad = \frac{1}{L(r, \alpha, s, \beta \,|\, x, t_x, T)} \int_0^\infty \int_0^\infty \lambda^l \mu^m L(\lambda, \mu \,|\, x, t_x, T) g(\lambda \,|\, r, \alpha) g(\mu \,|\, s, \beta)\, d\lambda\, d\mu \\
&\quad = \frac{1}{L(r, \alpha, s, \beta \,|\, x, t_x, T)} \\
&\qquad \times \int_0^\infty \int_0^\infty \lambda^l \mu^m L(\lambda, \mu \,|\, x, t_x, T) \frac{\alpha^r \lambda^{r-1} e^{-\lambda\alpha}}{\Gamma(r)} \frac{\beta^s \mu^{s-1} e^{-\mu\beta}}{\Gamma(s)}\, d\lambda\, d\mu \\
&\quad = \frac{1}{L(r, \alpha, s, \beta \,|\, x, t_x, T)} \\
&\qquad \times \int_0^\infty \int_0^\infty L(\lambda, \mu \,|\, x, t_x, T) \frac{\alpha^r \lambda^{r+l-1} e^{-\lambda\alpha}}{\Gamma(r)} \frac{\beta^s \mu^{s+m-1} e^{-\mu\beta}}{\Gamma(s)}\, d\lambda\, d\mu \\
&\quad = \frac{1}{L(r, \alpha, s, \beta \,|\, x, t_x, T)} \\
&\qquad \times \frac{\Gamma(r+l)\Gamma(s+m)}{\Gamma(r)\Gamma(s)\alpha^l \beta^m} \int_0^\infty \int_0^\infty L(\lambda, \mu \,|\, x, t_x, T) \frac{\alpha^{r+l} \lambda^{r+l-1} e^{-\lambda\alpha}}{\Gamma(r+l)} \frac{\beta^{s+m} \mu^{s+m-1} e^{-\mu\beta}}{\Gamma(s+m)}\, d\lambda\, d\mu \\
&\quad = \frac{\Gamma(r+l)\Gamma(s+m)}{\Gamma(r)\Gamma(s)\alpha^l \beta^m} \frac{L(r+l, \alpha, s+m, \beta \,|\, x, t_x, T)}{L(r, \alpha, s, \beta \,|\, x, t_x, T)}\,,
\end{aligned} \tag{8}$$

where $L(r+l, \alpha, s+m, \beta \,|\, x, t_x, T)$ is simply (2) evaluated using $r+l$ in place of r and $s+m$ in place of s.

[1] `http://functions.wolfram.com/HypergeometricFunctions/HypergeometricU/`

3 Marginal Posterior Distributions of the Latent Variables

3.1 Marginal Posterior of $\boldsymbol{\Lambda}$

Given the expression for the joint posterior distribution of Λ and M, (4), the marginal posterior distribution of Λ is given by:

$$\begin{aligned} g(\lambda \,|\, r, \alpha, s, \beta; x, t_x, T) &= \int_0^\infty g(\lambda, \mu \,|\, r, \alpha, s, \beta; x, t_x, T)\, d\mu \\ &= \frac{\mathsf{B}_1 + \mathsf{B}_2}{L(r, \alpha, s, \beta \,|\, x, t_x, T)}\,, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \mathsf{B}_1 &= \frac{\alpha^r \lambda^{r+x-1} e^{-\lambda(\alpha+t_x)}}{\Gamma(r)} \frac{\beta^s}{\Gamma(s)} \int_0^\infty \mu^s (\lambda+\mu)^{-1} e^{-\mu(\beta+t_x)} d\mu \\ &= \frac{\alpha^r \lambda^{r+x-2} e^{-\lambda(\alpha+t_x)}}{\Gamma(r)} \frac{\beta^s}{\Gamma(s)} \int_0^\infty \mu^s \big(1 + \frac{\mu}{\lambda}\big)^{-1} e^{-\mu(\beta+t_x)} d\mu \end{aligned}$$

which, recalling (7) (with $p = \beta + t_x$, $q = s + 1$, $v = 1$, and $a = 1/\lambda$),

$$= \frac{\alpha^r \lambda^{r+s+x-1} e^{-\lambda(\alpha+t_x)}}{\Gamma(r)} s\beta^s \Psi\big(s+1, s+1; \lambda(\beta+t_x)\big)\,, \tag{10}$$

and

$$\begin{aligned} \mathsf{B}_2 &= \frac{\alpha^r \lambda^{r+x} e^{-\lambda(\alpha+T)}}{\Gamma(r)} \frac{\beta^s}{\Gamma(s)} \int_0^\infty \mu^{s-1} (\lambda+\mu)^{-1} e^{-\mu(\beta+T)} d\mu \\ &= \frac{\alpha^r \lambda^{r+x-1} e^{-\lambda(\alpha+T)}}{\Gamma(r)} \frac{\beta^s}{\Gamma(s)} \int_0^\infty \mu^{s-1} \big(1 + \frac{\mu}{\lambda}\big)^{-1} e^{-\mu(\beta+T)} d\mu \end{aligned}$$

which, recalling (7) (with $p = \beta + T$, $q = s$, $v = 1$, and $a = 1/\lambda$),

$$= \frac{\alpha^r \lambda^{r+s+x-1} e^{-\lambda(\alpha+T)}}{\Gamma(r)} \beta^s \Psi\big(s, s; \lambda(\beta+T)\big)\,. \tag{11}$$

The mean of this distribution is simply (8) evaluated with $l = 1$ and $m = 0$.

3.2 Marginal Posterior of $\boldsymbol{M}$

Given the expression for the joint posterior distribution of Λ and M, (4), the marginal posterior distribution of the M is given by:

$$\begin{aligned} g(\mu \,|\, r, \alpha, s, \beta; x, t_x, T) &= \int_0^\infty g(\lambda, \mu \,|\, r, \alpha, s, \beta; x, t_x, T)\, d\lambda \\ &= \frac{\mathsf{C}_1 + \mathsf{C}_2}{L(r, \alpha, s, \beta \,|\, x, t_x, T)}\,, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \mathsf{C}_1 &= \frac{\beta^s \mu^s e^{-\mu(\beta+t_x)}}{\Gamma(s)} \frac{\alpha^r}{\Gamma(r)} \int_0^\infty \lambda^{r+x-1} (\lambda+\mu)^{-1} e^{-\lambda(\alpha+t_x)} d\lambda \\ &= \frac{\beta^s \mu^{s-1} e^{-\mu(\beta+t_x)}}{\Gamma(s)} \frac{\alpha^r}{\Gamma(r)} \int_0^\infty \lambda^{r+x-1} (1 + \frac{\lambda}{\mu})^{-1} e^{-\lambda(\alpha+t_x)} d\lambda \end{aligned}$$

which, recalling (7) (with $p = \alpha + t_x$, $q = r + x$, $v = 1$, and $a = 1/\mu$),

$$= \frac{\beta^s \mu^{r+s+x-1} e^{-\mu(\beta+t_x)}}{\Gamma(s)} \alpha^r \frac{\Gamma(r+x)}{\Gamma(r)} \Psi\big(r+x, r+x; \mu(\alpha+t_x)\big)\,, \tag{13}$$

and

$$\begin{aligned}
\mathsf{C}_2 &= \frac{\beta^s \mu^{s-1} e^{-\mu(\beta+T)}}{\Gamma(s)} \frac{\alpha^r}{\Gamma(r)} \int_0^\infty \lambda^{r+x} (\lambda+\mu)^{-1} e^{-\lambda(\alpha+T)} d\lambda \\
&= \frac{\beta^s \mu^{s-2} e^{-\mu(\beta+T)}}{\Gamma(s)} \frac{\alpha^r}{\Gamma(r)} \int_0^\infty \lambda^{r+x} (1+\frac{\lambda}{\mu})^{-1} e^{-\lambda(\alpha+T)} d\lambda
\end{aligned}$$

which, recalling (7) (with $p = \alpha + T$, $q = r + x + 1$, $v = 1$, and $a = 1/\mu$),

$$= \frac{\beta^s \mu^{r+s+x-1} e^{-\mu(\beta+T)}}{\Gamma(s)} \alpha^r \frac{\Gamma(r+x+1)}{\Gamma(r)} \Psi\big(r+x+1, r+x+1; \mu(\alpha+T)\big)\,. \tag{14}$$

The mean of this distribution is simply (8) evaluated with $l = 0$ and $m = 1$.

4 $P(\text{alive at } T+t \,|\, x, t_x, T)$

Standing at time T, what is the probability that a customer with purchase history (x, t_x, T) will still be "alive" at $T+t$? Let us first derive the conditional distribution of Ω, $F(\omega \,|\, r, \alpha, s, \beta; x, t_x, T)$.

Decomposing the structure of this quantity,

$$\begin{aligned}
&F(\omega \,|\, r, \alpha, s, \beta; x, t_x, T) \\
&\quad = \int_0^\infty \int_0^\infty F(\omega \,|\, \mu, \Omega > T) P(\Omega > T \,|\, \lambda, \mu; x, t_x, T) g(\lambda, \mu \,|\, r, \alpha, s, \beta; x, t_x, T)\, d\lambda\, d\mu\,.
\end{aligned} \tag{15}$$

By definition,

$$\begin{aligned}
F(\omega \,|\, \mu, \Omega > T) &= 1 - S(\omega \,|\, \mu, \Omega > T) \\
&= 1 - \frac{S(\omega \,|\, \mu)}{S(T \,|\, \mu)} \\
&= 1 - e^{-\mu(\omega - T)}\,,\ \omega > T.
\end{aligned} \tag{16}$$

Substituting (3)–(6) and (16) in (15) gives us

$$\begin{aligned}
&F(\omega \,|\, r, \alpha, s, \beta; x, t_x, T) \\
&\quad = 1 - \frac{1}{L(r, \alpha, s, \beta \,|\, x, t_x, T)} \int_0^\infty \int_0^\infty e^{-\mu(\omega-T)} \lambda^x e^{-(\lambda+\mu)T} g(\lambda \,|\, r, \alpha) g(\mu \,|\, s, \beta)\, d\lambda\, d\mu \\
&\quad = 1 - \frac{1}{L(r, \alpha, s, \beta \,|\, x, t_x, T)} \bigg\{ \int_0^\infty \lambda^x e^{-\lambda T} g(\lambda \,|\, r, \alpha)\, d\lambda \bigg\} \bigg\{ \int_0^\infty e^{-\mu\omega} g(\mu \,|\, s, \beta) d\mu \bigg\} \\
&\quad = 1 - \frac{\Gamma(r+x)}{\Gamma(r)} \left(\frac{\alpha}{\alpha+T}\right)^r \left(\frac{1}{\alpha+T}\right)^x \left(\frac{\beta}{\beta+\omega}\right)^s \Big/ L(r, \alpha, s, \beta \,|\, x, t_x, T)\,,\ \omega > T\,.
\end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned}
&P(\text{alive at } T+t \,|\, r, \alpha, s, \beta; x, t_x, T) \\
&\quad = 1 - F(T+t \,|\, r, \alpha, s, \beta; x, t_x, T) \\
&\quad = \frac{\Gamma(r+x)}{\Gamma(r)} \left(\frac{\alpha}{\alpha+T}\right)^r \left(\frac{1}{\alpha+T}\right)^x \left(\frac{\beta}{\beta+T+t}\right)^s \Big/ L(r, \alpha, s, \beta \,|\, x, t_x, T)\,.
\end{aligned} \tag{18}$$

References

Fader, Peter S. and Bruce G. S. Hardie (2005), "A Note on Deriving the Pareto/NBD Model and Related Expressions," <http://brucehardie.com/notes/009/>.

Gradshteyn, I. S. and I. M. Ryzhik (1994), *Table of Integrals, Series, and Products*, 5th edition, San Diego, CA: Academic Press.