

# Deriving an Expression for $P(X(t, t + \tau) = x)$ Under the Pareto/NBD Model

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## 1 Introduction

Schmittlein et al. (1987) and Fader and Hardie (2006) derive expressions for  $P(X(t) = x)$ , where the random variable  $X(t)$  denotes the number of transactions observed in the time interval  $(0, t]$ , as implied by the Pareto/NBD model assumptions. In this note, we derive the corresponding expression for  $P(X(t, t + \tau) = x)$ , where the random variable  $X(t, t + \tau)$  denotes the number of transactions observed in the time interval  $(t, t + \tau]$ .

In Section 2 we review the assumptions underlying the Pareto/NBD model. In Section 3, we derive an expression for  $P(X(t, t + \tau) = x)$  conditional on the unobserved latent characteristics  $\lambda$  and  $\mu$ ; this conditioning is removed in Section 4.

## 2 Model Assumptions

The Pareto/NBD model is based on the following assumptions:

- i. Customers go through two stages in their “lifetime” with a specific firm: they are “alive” for some period of time, then become permanently inactive.
- ii. While alive, the number of transactions made by a customer follows a Poisson process with transaction rate  $\lambda$ . This implies that the probability of observing  $x$  transactions in the time interval  $(0, t]$  is given by

$$P(X(t) = x | \lambda) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \dots$$

It also implies that, assuming the customer is alive through the time interval  $(t_a, t_b]$ ,

$$P(X(t_a, t_b) = x | \lambda) = \frac{[\lambda(t_b - t_a)]^x e^{-\lambda(t_b - t_a)}}{x!}, \quad x = 0, 1, 2, \dots$$

- iii. A customer’s unobserved “lifetime” of length  $\omega$  (after which he is viewed as being inactive) is exponentially distributed with dropout rate  $\mu$ :

$$f(\omega | \mu) = \mu e^{-\mu \omega}.$$

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- iv. Heterogeneity in transaction rates across customers follows a gamma distribution with shape parameter  $r$  and scale parameter  $\alpha$ :

$$g(\lambda | r, \alpha) = \frac{\alpha^r \lambda^{r-1} e^{-\lambda\alpha}}{\Gamma(r)}. \quad (1)$$

- v. Heterogeneity in dropout rates across customers follows a gamma distribution with shape parameter  $s$  and scale parameter  $\beta$ :

$$g(\mu | s, \beta) = \frac{\beta^s \mu^{s-1} e^{-\mu\beta}}{\Gamma(s)}. \quad (2)$$

- vi. The transaction rate  $\lambda$  and the dropout rate  $\mu$  vary independently across customers.

### 3 $P(X(t, t + \tau) = x)$ Conditional on $\lambda$ and $\mu$

Suppose we know an individual's unobserved latent characteristics  $\lambda$  and  $\mu$ . For  $x > 0$ , there are two ways  $x$  purchases could have occurred in the interval  $(t, t + \tau]$ :

- i. The individual was alive at  $t$  and remained alive through the whole interval; this occurs with probability  $e^{-\mu(t+\tau)}$ . The probability of the individual making  $x$  purchases, given that he was alive during the whole interval, is  $(\lambda\tau)^x e^{-\lambda\tau} / x!$ . It follows that the probability of remaining alive through the interval  $(t, t + \tau]$  and making  $x$  purchases is

$$\frac{(\lambda\tau)^x e^{-\lambda\tau} e^{-\mu(t+\tau)}}{x!}. \quad (3)$$

- ii. The individual was alive at  $t$  but died at some point  $\omega$  ( $< t + \tau$ ), making  $x$  purchases in the interval  $(t, \omega]$ . The probability of this occurring is

$$\begin{aligned} & \int_t^{t+\tau} \frac{[\lambda(\omega - t)]^x e^{-\lambda(\omega-t)}}{x!} \mu e^{-\mu\omega} d\omega \\ &= e^{-\mu t} \lambda^x \mu \int_t^{t+\tau} \frac{(\omega - t)^x e^{-(\lambda+\mu)(\omega-t)}}{x!} d\omega \\ &= e^{-\mu t} \lambda^x \mu \int_0^\tau \frac{s^x e^{-(\lambda+\mu)s}}{x!} ds \\ &= e^{-\mu t} \frac{\lambda^x \mu}{(\lambda + \mu)^{x+1}} \int_0^\tau \frac{(\lambda + \mu)^{x+1} s^x e^{-(\lambda+\mu)s}}{x!} ds \end{aligned}$$

which, noting that the integrand is an Erlang- $(x + 1)$  pdf,

$$= e^{-\mu t} \left( \frac{\lambda}{\lambda + \mu} \right)^x \left( \frac{\mu}{\lambda + \mu} \right) \left[ 1 - e^{-(\lambda+\mu)\tau} \sum_{i=0}^x \frac{[(\lambda + \mu)\tau]^i}{i!} \right]. \quad (4)$$

These two scenarios also hold for the case of  $x = 0$  but need to be augmented by an additional reason as to why no purchases could have occurred in the interval  $(t, t + \tau]$ : the individual was dead at the beginning of the interval, which occurs with probability

$$1 - e^{-\mu t}. \quad (5)$$

Combining (3)–(5) gives us the following expression for the probability of observing  $x$  purchases in the interval  $(t, t + \tau]$ , conditional on  $\lambda$  and  $\mu$ :

$$\begin{aligned} P(X(t, t + \tau) = x | \lambda, \mu) &= \delta_{x=0} \left[ 1 - e^{-\mu t} \right] + \frac{(\lambda \tau)^x e^{-\lambda \tau} e^{-\mu(t+\tau)}}{x!} \\ &\quad + \left( \frac{\lambda}{\lambda + \mu} \right)^x \left( \frac{\mu}{\lambda + \mu} \right) e^{-\mu t} \\ &\quad - \left( \frac{\lambda}{\lambda + \mu} \right)^x \left( \frac{\mu}{\lambda + \mu} \right) e^{-\lambda \tau} e^{-\mu(t+\tau)} \sum_{i=0}^x \frac{[(\lambda + \mu)\tau]^i}{i!}. \end{aligned} \quad (6)$$

## 4 Removing the Conditioning on $\lambda$ and $\mu$

In reality, we never know an individual's latent characteristics; we therefore remove the conditioning on  $\lambda$  and  $\mu$  by taking the expectation of (6) over the distributions of  $\Lambda$  and  $M$ :

$$\begin{aligned} P(X(t, t + \tau) = x | r, \alpha, s, \beta) \\ = \int_0^\infty \int_0^\infty P(X(t, t + \tau) = x | \lambda, \mu) g(\lambda | r, \alpha) g(\mu | s, \beta) d\lambda d\mu. \end{aligned} \quad (7)$$

Substituting (1), (2), and (6) in (7) gives us

$$P(X(t, t + \tau) = x | r, \alpha, s, \beta) = \delta_{x=0} \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 - \sum_{i=0}^x \frac{\tau^i}{i!} \mathbf{A}_4 \quad (8)$$

where

$$\mathbf{A}_1 = \int_0^\infty \left[ 1 - e^{-\mu t} \right] g(\mu | s, \beta) d\mu \quad (9)$$

$$\mathbf{A}_2 = \int_0^\infty \int_0^\infty \frac{(\lambda \tau)^x e^{-\lambda \tau} e^{-\mu(t+\tau)}}{x!} g(\lambda | r, \alpha) g(\mu | s, \beta) d\lambda d\mu \quad (10)$$

$$\mathbf{A}_3 = \int_0^\infty \int_0^\infty \left( \frac{\lambda}{\lambda + \mu} \right)^x \left( \frac{\mu}{\lambda + \mu} \right) e^{-\mu t} g(\lambda | r, \alpha) g(\mu | s, \beta) d\lambda d\mu \quad (11)$$

$$\mathbf{A}_4 = \int_0^\infty \int_0^\infty \left( \frac{\lambda}{\lambda + \mu} \right)^x \left( \frac{\mu}{\lambda + \mu} \right) (\lambda + \mu)^i e^{-\lambda \tau} e^{-\mu(t+\tau)} g(\lambda | r, \alpha) g(\mu | s, \beta) d\lambda d\mu \quad (12)$$

Solving (9) and (10) is trivial:

$$\mathbf{A}_1 = 1 - \left( \frac{\beta}{\beta + t} \right)^s \quad (13)$$

$$\mathbf{A}_2 = \frac{\Gamma(r+x)}{\Gamma(r)x!} \left( \frac{\alpha}{\alpha + \tau} \right)^r \left( \frac{\tau}{\alpha + \tau} \right)^x \left( \frac{\beta}{\beta + t + \tau} \right)^s \quad (14)$$

To solve (11), consider the transformation  $Y = M/(\Lambda + M)$  and  $Z = \Lambda + M$ . Using the transformation technique (Casella and Berger 2002, Section 4.3, pp. 156–162; Mood et al. 1974, Section 6.2, p. 204ff), it follows that the joint distribution of  $Y$  and  $Z$  is

$$g(y, z | \alpha, \beta, r, s) = \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} y^{s-1} (1-y)^{r-1} z^{r+s-1} e^{-z(\alpha - (\alpha - \beta)y)}. \quad (15)$$

Noting that the inverse of this transformation is  $\lambda = (1 - y)z$  and  $\mu = yz$ , it follows that

$$\begin{aligned}
A_3 &= \int_0^1 \int_0^\infty y(1-y)^x e^{-yzt} g(y, z | \alpha, \beta, r, s) dz dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 \int_0^\infty y^s (1-y)^{r+x-1} z^{r+s-1} e^{-z(\alpha - (\alpha - (\beta+t))y)} dz dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 y^s (1-y)^{r+x-1} \left\{ \int_0^\infty z^{r+s-1} e^{-z(\alpha - (\alpha - (\beta+t))y)} dz \right\} dy \\
&= \alpha^r \beta^s \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 y^s (1-y)^{r+x-1} (\alpha - (\alpha - (\beta+t))y)^{-(r+s)} dy \\
&= \frac{1}{B(r, s)} \frac{\alpha^r \beta^s}{\alpha^{r+s}} \int_0^1 y^s (1-y)^{r+x-1} \left[ 1 - \left( \frac{\alpha - (\beta+t)}{\alpha} \right) y \right]^{-(r+s)} dy
\end{aligned}$$

which, recalling Euler's integral for the Gaussian hypergeometric function,<sup>1</sup>

$$= \left( \frac{\beta}{\alpha} \right)^s \frac{B(r+x, s+1)}{B(r, s)} {}_2F_1(r+s, s+1; r+s+x+1; \frac{\alpha - (\beta+t)}{\alpha}). \quad (16)$$

Looking closely at (16), we see that the argument of the Gaussian hypergeometric function,  $\frac{\alpha - (\beta+t)}{\alpha}$ , is guaranteed to be bounded between 0 and 1 when  $\alpha \geq \beta + t$ , thus ensuring convergence of the series representation of the function. However, when  $\alpha < \beta + t$  we can be faced with the situation where  $\frac{\alpha - (\beta+t)}{\alpha} < -1$ , in which case the series is divergent.

Applying the linear transformation (Abramowitz and Stegun 1972, equation 15.3.4)

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1(a, c-b; c; \frac{z}{z-1}), \quad (17)$$

gives us

$$A_3 = \frac{\alpha^r \beta^s}{(\beta+t)^{r+s}} \frac{B(r+x, s+1)}{B(r, s)} {}_2F_1(r+s, r+x; r+s+x+1; \frac{\beta+t-\alpha}{\beta+t}). \quad (18)$$

We note that the argument of the above Gaussian hypergeometric function is bounded between 0 and 1 when  $\alpha \leq \beta + t$ . We therefore present (16) and (18) as solutions to (11), using (16) when  $\alpha \geq \beta + t$  and (18) when  $\alpha \leq \beta + t$ . We can write this as

$$A_3 = \alpha^r \beta^s \frac{B(r+x, s+1)}{B(r, s)} B_1 \quad (19)$$

where

$$B_1 = \begin{cases} {}_2F_1(r+s, s+1; r+s+x+1; \frac{\alpha - (\beta+t)}{\alpha}) / \alpha^{r+s} & \text{if } \alpha \geq \beta + t \\ {}_2F_1(r+s, r+x; r+s+x+1; \frac{\beta+t-\alpha}{\beta+t}) / (\beta+t)^{r+s} & \text{if } \alpha \leq \beta + t \end{cases} \quad (20)$$

To solve (12), we also make use of the transformation  $Y = M/(\Lambda + M)$  and  $Z = \Lambda + M$ . Given (15), it follows that

$$\begin{aligned}
A_4 &= \int_0^1 \int_0^\infty y(1-y)^x z^i e^{-(1-y)z\tau} e^{-yz(t+\tau)} g(y, z | \alpha, \beta, r, s) dz dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 \int_0^\infty y^s (1-y)^{r+x-1} z^{r+s+i-1} e^{-z(\alpha + \tau - (\alpha - (\beta+t))y)} dz dy \\
&= \frac{\alpha^r \beta^s}{\Gamma(r)\Gamma(s)} \int_0^1 y^s (1-y)^{r+x-1} \left\{ \int_0^\infty z^{r+s+i-1} e^{-z(\alpha + \tau - (\alpha - (\beta+t))y)} dz \right\} dy \\
&= \alpha^r \beta^s \frac{\Gamma(r+s+i)}{\Gamma(r)\Gamma(s)} \int_0^1 y^s (1-y)^{r+x-1} (\alpha + \tau - (\alpha - (\beta+t))y)^{-(r+s+i)} dy \\
&= \frac{\Gamma(r+s+i)}{\Gamma(r)\Gamma(s)} \frac{\alpha^r \beta^s}{(\alpha + \tau)^{r+s+i}} \int_0^1 y^s (1-y)^{r+x-1} \left[ 1 - \left( \frac{\alpha - (\beta+t)}{\alpha + \tau} \right) y \right]^{-(r+s+i)} dy
\end{aligned}$$

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<sup>1</sup> ${}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c > b.$

which, recalling Euler's integral for the Gaussian hypergeometric function,

$$= \frac{\Gamma(r+s+i)}{\Gamma(r+s)} \frac{\alpha^r \beta^s}{(\alpha+\tau)^{r+s+i}} \frac{B(r+x, s+1)}{B(r, s)} \\ \times {}_2F_1\left(r+s+i, s+1; r+s+x+1; \frac{\alpha-(\beta+t)}{\alpha+\tau}\right). \quad (21)$$

Noting that the argument of the Gaussian hypergeometric function is only guaranteed to be bounded between 0 and 1 when  $\alpha \geq \beta+t$  ( $\forall \tau > 0$ ), we apply the linear transformation (17), which gives us

$$A_4 = \frac{\Gamma(r+s+i)}{\Gamma(r+s)} \frac{\alpha^r \beta^s}{(\beta+t+\tau)^{r+s+i}} \frac{B(r+x, s+1)}{B(r, s)} \\ \times {}_2F_1\left(r+s+i, r+x; r+s+x+1; \frac{\beta+t-\alpha}{\beta+t+\tau}\right), \quad (22)$$

The argument of the above Gaussian hypergeometric function is bounded between 0 and 1 when  $\alpha \leq \beta+t$  ( $\forall \tau > 0$ ). We therefore present (21) and (22) as solutions to (12): we use (21) when  $\alpha \geq \beta+t$  and (22) when  $\alpha \leq \beta+t$ . We can write this as

$$A_4 = \alpha^r \beta^s \frac{\Gamma(r+s+i)}{\Gamma(r+s)} \frac{B(r+x, s+1)}{B(r, s)} B_2 \quad (23)$$

where

$$B_2 = \begin{cases} {}_2F_1\left(r+s+i, s+1; r+s+x+1; \frac{\alpha-(\beta+t)}{\alpha+\tau}\right) / (\alpha+\tau)^{r+s+i} & \text{if } \alpha \geq \beta+t \\ {}_2F_1\left(r+s+i, r+x; r+s+x+1; \frac{\beta+t-\alpha}{\beta+t+\tau}\right) / (\beta+t+\tau)^{r+s+i} & \text{if } \alpha \leq \beta+t \end{cases} \quad (24)$$

Substituting (13), (14), (19), and (23) in (8), and noting that

$$\frac{\Gamma(r+s+i)}{\Gamma(r+s)} \frac{1}{i!} = \frac{1}{iB(r+s, i)},$$

gives us the following expression for the probability of observing  $x$  transactions in the time interval  $(t, t+\tau]$ :

$$P(X(t, t+\tau) = x | r, \alpha, s, \beta) \\ = \delta_{x=0} \left[ 1 - \left( \frac{\beta}{\beta+t} \right)^s \right] + \frac{\Gamma(r+x)}{\Gamma(r)x!} \left( \frac{\alpha}{\alpha+\tau} \right)^r \left( \frac{\tau}{\alpha+\tau} \right)^x \left( \frac{\beta}{\beta+t+\tau} \right)^s \\ + \alpha^r \beta^s \frac{B(r+x, s+1)}{B(r, s)} \left\{ B_1 - \sum_{i=0}^x \frac{\tau^i}{iB(r+s, i)} B_2 \right\} \quad (25)$$

where expressions for  $B_1$  and  $B_2$  are given in (20) and (24), respectively.

We note that for  $t = 0$ , (25) reduces to the implied expression for  $P(X(\tau) = x)$  as given in Fader and Hardie (2006, equation 16).

## References

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